Note two trace class operator inequalities

Ming-Hua Lin
College of Mathematics and Information Science,
Shaanxi Normal University, Xi'an 710062, P. R. China
E-mail: miwalin@163.com

Abstract

In this note, we generalize the inequality about the trace of positive semidefinite matrix

$$trAB \le (trA^p)^{\frac{1}{p}}(trB^q)^{\frac{1}{q}}$$

and

$$(tr(A+B)^p)^{\frac{1}{p}} \le (trA^p)^{\frac{1}{p}} + (trB^p)^{\frac{1}{p}},$$

which are due to J.R.Magnus, to Hilbert space, and obtain several relevant inequalities about positive trace class operator.

Key Words: Hilbert space, Hölder's inequality, Minkowsky's inequality.

1 Introduction

Since R.Bellman posed two conjectures on the matrix trace inequality in 1980, many authors have been discussing and gaining inequalities of this type. Matrix can be regarded as an operator in finite Hilbert space, so generalizing matrix trace inequalities to trace class operator inequalities must be meaningful. For related works the reader can refer to [1] and [2]. In this paper, we generalize two matrix trace inequalities to Hilbert space, and obtain several relevant inequalities about positive trace class operator. Denote by $B(H)$ the set of all bound linear operators of an infinite separable Hilbert space. Let $A \in B(H)$ be a compact operator. If $\sum_{i=1}^{\infty} s_i(A) \le \infty$, where $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A) \ge \cdots$ are the singular values of A, that is, the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$, then A is called a trace class operator. The trace of A is denoted by trA and the set of all trace class operator is denoted by $B_1(H)$. Let $A \in B_1(H)$, then $\|A\|_1 = \sum_{i=1}^{\infty} s_i(A)$ is called the trace norm of A. We know that if A is positive in $B_1(H)$, then $A = \sum_{i=1}^{\infty} s_i(A) e_i \otimes \overline{e_i}$, where e_i is the relevant eigenvector of $s_i(A)$.

2 Lemmas

Lemma 2.1 [3] *Let A, B are positive semidefinte Hermitians. Then*

$$trAB \le (trA^p)^{\frac{1}{p}}(trB^q)^{\frac{1}{q}}, \tag{2.1}$$

where $\frac{1}{p}+\frac{1}{q}=1, p>1$.

Lemma 2.2 *If* $A_i, B_i, i=1,\cdots,k$ *are positive semidefinite Hermitians, then*

$$tr\sum_{i=1}^{k}A_iB_i \le (tr\sum_{i=1}^{k}A_i^p)^{\frac{1}{p}}(tr\sum_{i=1}^{k}B_i^q)^{\frac{1}{q}}, \tag{2.2}$$

where $\frac{1}{p}+\frac{1}{q}=1, p>1$

Proof. *By Lemma 2.1,* $tr\sum_{i=1}^{k}A_iB_i = \sum_{i=1}^{k}trA_iB_i \le \sum_{i=1}^{k}(trA_i^p)^{\frac{1}{p}}(trB_i^q)^{\frac{1}{q}} \le (\sum_{i=1}^{k}trA_i^p)^{\frac{1}{p}}(\sum_{i=1}^{k}trB_i^q)^{\frac{1}{q}} = (tr\sum_{i=1}^{k}A_i^p)^{\frac{1}{p}}(tr\sum_{i=1}^{k}B_i^q)^{\frac{1}{q}}$. □

Lemma 2.3 [4] *Let* $S\in B(H)$ *and* $T\in B_1(H)$. *Then* $\|TS\|_1 \le \|T\|_1\|S\|$ *and* $\|ST\|_1 \le \|S\|\|T\|_1$.

Lemma 2.4 [3] *Let A, B are positive semidefinte Hermitians. Then*

$$(tr(A+B)^p)^{\frac{1}{p}} \le (trA^p)^{\frac{1}{p}} + (trB^p)^{\frac{1}{p}}, \tag{2.3}$$

where $p>1$.

Lemma 2.5 *Let* $A_i, B_i, i=1,\cdots,k$ *are positive semidefinte Hermitians. Then*

$$(tr\sum_{i=1}^{k}(A_i+B_i)^p)^{\frac{1}{p}} \le (tr\sum_{i=1}^{k}A_i^p)^{\frac{1}{p}} + (tr\sum_{i=1}^{k}B_i^p)^{\frac{1}{q}}, \tag{2.4}$$

where $p>1$.

Proof. *By Lemma 2.4,* $(tr\sum_{i=1}^{k}(A_i+B_i)^p)^{\frac{1}{p}} = (\sum_{i=1}^{k}tr(A_i+B_i)^p)^{\frac{1}{p}} \le$
$(\sum_{i=1}^{k}((trA_i^p)^{\frac{1}{p}}+(trB_i^p)^{\frac{1}{p}})^p)^{\frac{1}{p}} \le (\sum_{i=1}^{k}trA_i^p)^{\frac{1}{p}} + (\sum_{i=1}^{k}trB_i^p)^{\frac{1}{p}}$
$= (tr\sum_{i=1}^{k}A_i^p)^{\frac{1}{p}} + (tr\sum_{i=1}^{k}B_i^p)^{\frac{1}{q}}$ □

3 Main results

Theorem 3.1 *If* $A, B\in B_1(H)$ *are positive, then*

$$trAB \le (trA^p)^{\frac{1}{p}}(trB^q)^{\frac{1}{q}}, \tag{3.1}$$

where $\frac{1}{p}+\frac{1}{q}=1, p>1$.

Proof. *Let* $A=\sum_{i=1}^{\infty}s_i(A)e_i\otimes\overline{e_i}$ *and* $B=\sum_{i=1}^{\infty}s_i(B)g_i\otimes\overline{g_i}$. *We write* $A_n=\sum_{i=1}^{n}s_i(A)e_i\otimes\overline{e_i}$, $B_n=\sum_{i=1}^{n}s_i(B)g_i\otimes\overline{g_i}$ *and* $M_n=span\{e_1,\cdots,e_n, g_1,\cdots,g_n\}$. *Let* P_n *be the projection from H to* M_n, *and* $\overline{A_n}=P_nAP_n, \overline{B_n}=P_nBP_n$. *Obviously,* $\overline{A_n}$ *and* $\overline{B_n}$ *are positive semidefinite matrices over* $\mathbb{C}$ *and*

$\|\overline{A_n}\| \leq \|A\|, \|\overline{B_n}\| \leq \|B\|$. *Since* $\|e_i \otimes \overline{e_i}\|_1 = 1$, *by Lemma 3, we have* $\|A - \overline{A_n}\|_1 \leq \|A - A_n\|_1 + \|A_n - \overline{A_n}\|_1 = \|\sum_{i=n+1}^{n} s_i(A) e_i \otimes \overline{e_i}\|_1 + \|\sum_{i=n+1}^{n} s_i(A) P_n(e_i \otimes \overline{e_i}) P_n\|_1 \leq \sum_{i=n+1}^{n} s_i(A) + \sum_{i=n+1}^{n} s_i(A) = 2\sum_{i=n+1}^{n} s_i(A) \longrightarrow 0, (n \longrightarrow \infty)$

Since $|trA - tr\overline{A_n}| \leq \|A - \overline{A_n}\|_1 \longrightarrow 0, (n \longrightarrow \infty)$, *we have* $|trA^p - tr\overline{A_n}^p| \longrightarrow 0, (n \longrightarrow \infty)$, *and so* $|(trA^p)^{\frac{1}{p}} - (tr\overline{A_n}^p)^{\frac{1}{p}}| \longrightarrow 0, (n \longrightarrow \infty)$.

Similarly, $|(trB^p)^{\frac{1}{p}} - (tr\overline{B_n}^p)^{\frac{1}{p}}| \longrightarrow 0, (n \longrightarrow \infty)$.

Thus $|(trA^p)^{\frac{1}{p}}(trB^p)^{\frac{1}{p}} - (tr\overline{A_n}^p)^{\frac{1}{p}}(tr\overline{B_n}^p)^{\frac{1}{p}}| \leq (trA^p)^{\frac{1}{p}}|(trB^p)^{\frac{1}{p}} - (tr\overline{B_n}^p)^{\frac{1}{p}}| + (tr\overline{B_n}^p)^{\frac{1}{p}}|(trA^p)^{\frac{1}{p}} - (tr\overline{A_n}^p)^{\frac{1}{p}}| \longrightarrow 0, (n \longrightarrow \infty)$, *i.e.,*

$$(tr\overline{A_n}^p)^{\frac{1}{p}}(tr\overline{B_n}^p)^{\frac{1}{p}} \longrightarrow (trA^p)^{\frac{1}{p}}(trB^p)^{\frac{1}{p}}, (n \longrightarrow \infty). \tag{3.2}$$

Since $\|AB - \overline{A_n}\overline{B_n}\|_1 \leq \|A\| \, \|B - \overline{B_n}\|_1 + \|A - \overline{A_n}\|_1 \|B\| \longrightarrow 0, (n \longrightarrow \infty)$. *We see that*

$$tr\overline{A_n} tr\overline{B_n} \longrightarrow trAB (n \longrightarrow \infty). \tag{3.3}$$

By Lemma 2.1, combining (3.2) and (3.3), we get $trAB \leq (trA^p)^{\frac{1}{p}}(trB^q)^{\frac{1}{q}}$. *This completes the proof.* □

Theorem 3.2 *If* $A_i, B_i, i = 1, \cdots, k \in B_1(H)$ *are positive, then*

$$tr\sum_{i=1}^{k} A_i B_i \leq (tr\sum_{i=1}^{k} A_i^p)^{\frac{1}{p}}(tr\sum_{i=1}^{k} B_i^q)^{\frac{1}{q}}, \tag{3.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 1$.

Proof. *Let* $A_{in} = \sum_{j=1}^{n} s_j(A_i) e_{ij} \otimes \overline{e_{ij}}$, $B_n = \sum_{j=1}^{n} s_j(B) g_{ij} \otimes \overline{g_{ij}}$ *and* $M_{in} = span\{e_{i1}, \cdots, e_{in}, g_{i1}, \cdots, g_{in}\}, i = 1, \cdots, k$. *Let* P_{in} *be the projection from* H *to* M_{in}, *and* $\overline{A_{in}} = P_{in} A_i P_{in}, \overline{B_{in}} = P_{in} B_i P_{in}$. *By theorem 3.1, it suffices to show* $\sum_{i=1}^{k}(trA_i^p)^{\frac{1}{p}}(trB_i^q)^{\frac{1}{q}} \leq (tr\sum_{i=1}^{k} A_i^p)^{\frac{1}{p}}(tr\sum_{i=1}^{k} B_i^q)^{\frac{1}{q}}$. *And it is easy to find that* $\sum_{i=1}^{k}(tr\overline{A_{in}}^p)^{\frac{1}{p}}(tr\overline{B_{in}}^q)^{\frac{1}{q}} \longrightarrow \sum_{i=1}^{k}(trA_i^p)^{\frac{1}{p}}(trB_i^q)^{\frac{1}{q}}$ *and* $(tr\sum_{i=1}^{k}\overline{A_{in}}^p)^{\frac{1}{p}}(tr\sum_{i=1}^{k}\overline{B_{in}}^q)^{\frac{1}{q}} \longrightarrow (tr\sum_{i=1}^{k} A_i^p)^{\frac{1}{p}}(tr\sum_{i=1}^{k} B_i^q)^{\frac{1}{q}}, (n \longrightarrow \infty)$. *Thus the proof is complete.* □

In almost a similar manner, we can show the following theorems.

Theorem 3.3 *Let* $A, B \in B_1(H)$ *are positive. Then*

$$(tr(A+B)^p)^{\frac{1}{p}} \leq (trA^p)^{\frac{1}{p}} + (trB^p)^{\frac{1}{p}}, \tag{3.5}$$

where $p > 1$.

Theorem 3.4 *Let* $A_i, B_i, i = 1, \cdots, k \in B_1(H)$ *are positive semidefinte Hermitians. Then*

$$(tr\sum_{i=1}^{k}(A_i + B_i)^p)^{\frac{1}{p}} \leq (tr\sum_{i=1}^{k} A_i^p)^{\frac{1}{p}} + (tr\sum_{i=1}^{k} B_i^p)^{\frac{1}{q}}, \tag{3.6}$$

where $p > 1$.

References

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一个控制不等式

定理：非负向量 $a^{\downarrow}=(a_1,\cdots,a_n)$, $b^{\downarrow}=(b_1,\cdots,b_n)$, $\lambda^{\downarrow}=(\lambda_1,\cdots,\lambda_n)$, $\eta^{\downarrow}=(\eta_1,\cdots,\eta_n)$.

若 $\lambda \succ a$, $\eta \succ b$, 则 $\lambda\cdot\eta \succ_w a\cdot b$.

证明　先证明 $\lambda\cdot\eta \succ_w a\cdot\eta$, 即 $\sum_{i=1}^{k}\lambda_i\eta_i \geq \sum_{i=1}^{k}a_i\eta_i$, $k=1,\cdots,n$.

即证 $\sum_{i=1}^{k}(\lambda_i-a_i)\eta_i \geq 0$. 由Abel变换 $\sum_{i=1}^{k}u_iv_i=\sum_{i=1}^{k-1}(v_i-v_{i+1})U_i+v_kU_k$, 其中 $U_i=\sum_{j=1}^{i}u_j$

令 $v_i=\eta_i$, $u_i=\lambda_i-a_i$, 由于 $\lambda\succ a$ 知 $U_i\geq 0$, 且 η_i 递减非负, 故 $\sum_{i=1}^{k}(\lambda_i-a_i)\eta_i\geq 0$ $k=1,\cdots,n$.

同样的方法可证 $a\cdot\eta \succ_w a\cdot b$ 于是 $\lambda\cdot\eta \succ_w a\cdot b$.

利用这个定理可以证明，当n为偶数时 $trA^{n}\leq$ [illegible]

当n为奇数时呢？