§1.6. Exponential Form

Let r and θ be polar coordinates of the point (x, y) that corresponds to a nonzero complex number $z = x + iy$. Since $x = r\cos\theta$ and $y = r\sin\theta$, the number z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta). \tag{1.6.1}$$

If $z = 0$, the coordinate θ is undefined; and so it is always understood that $z \neq 0$ whenever $\arg z$ or $\mathrm{Arg}z$ defined below is discussed.

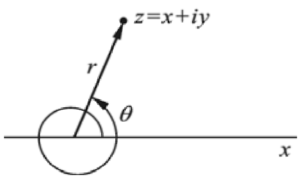

Fig. 1-6

In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z; that is, $r = |z|$. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector, see Fig. 1-6.

As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Those values can be determined from the equation

$$\tan\theta = y / x,$$

where the quadrant containing the *argument* of z, and the set of all such values is denoted by $\mathrm{Arg}z$. The principal value of $\mathrm{Arg}z$, denoted by $\arg z$, is the unique value Θ such that $-\pi < \Theta \le \pi$. Note that

$$\mathrm{Arg}z = \{\arg z + 2n\pi : n = 0, \pm 1, \pm 2, \ldots\},$$

simply, we write

$$\mathrm{Arg}z = \arg z + 2n\pi (n = 0, \pm 1, \pm 2, \ldots). \tag{1.6.2}$$

Also, when z is a negative real number, $\arg z$ has value π, not $-\pi$.

Example 1. The complex number $-1-i$, which lies in the third quadrant, has principal argument $-3\pi/4$. That is, $\arg(-1-i) = -\dfrac{3\pi}{4}$.

It must be emphasized that, because of the restriction $-\pi < \Theta \le \pi$ of the principal argument Θ, it is not true that $\arg(-1-i) = 5\pi/4$.

According to equation (1.6.2), we have

$$\mathrm{Arg}(-1-i) = -\frac{3\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Note that the term $\arg z$ on the right-hand side of equation (1.6.2) can be replaced by any particular value of $\mathrm{Arg}z$ and that one can write, for instance,

$$\mathrm{Arg}(-1-i) = \frac{5\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

The symbol $e^{i\theta}$, or $\exp(i\theta)$, is defined by means of *Euler's formula* as

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1.6.3}$$

where θ is to be measured in radians. It enables us to write the polar form (1.6.1) more compactly in *exponential form* as

$$z = re^{i\theta} \tag{1.6.4}$$

The choice of the symbol $e^{i\theta}$ will be fully motivated later on is Sec.2.8. Its use in Sec. 1.7 will, however, suggest that it is a natural choice.

Example 2. The number $-1-i$ in Example 1 has exponential form

$$-1-i=\sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}\right)\right]. \qquad (1.6.5)$$

Since $e^{-i\theta}=e^{i(-\theta)}$, this can also be written $-1-i=\sqrt{2}e^{-i3\pi/4}$. Expression (1.6.5) is, of course, only one of an infinite number of possibilities for the exponential form of $-1-i$:

$$-1-i=\sqrt{2}\exp\left[i\left(-\frac{3\pi}{4}+2n\pi\right)\right] \quad (n=0,\pm1,\pm2,\ldots). \qquad (1.6.6)$$

Note how expression (1.6.4) with $r=1$ tells us that the numbers $e^{i\theta}$ lie on the circle centered at the origin with radius unity, as shown in Fig. 1-7. Values of $e^{i\theta}$ are, then, immediate from that figure, without reference to Euler's formula. It is, for instance, geometrically obvious that $e^{i\pi}=-1, e^{i\pi/2}=i,$ and $e^{-14\pi i}=1$.

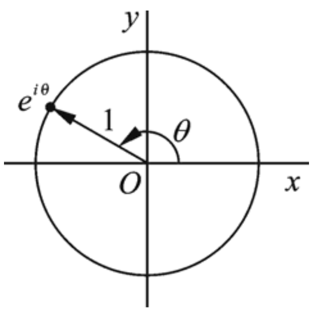

Fig. 1-7

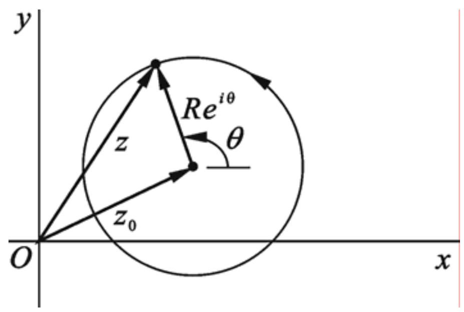

Fig.1-8

Note, too, that the equation

$$z=Re^{i\theta} \quad (0\le\theta\le2\pi) \qquad (1.6.7)$$

is a parametric representation of the circle $|z|=R$, centered at the origin with radius R. As the parameter θ increases from $\theta=0$ to $\theta=2\pi$, the point z starts from the positive real axis and traverses the circle once in the counterclockwise direction. More generally, the circle $|z-z_0|=R$, whose center is z_0 and whose radius is R, has the parametric representation

$$z=z_0+Re^{i\theta} \quad (0\le\theta\le2\pi). \qquad (1.6.8)$$

This can be seen vectorially (Fig. 1-8) by noting that a point z traversing the circle

$$|z-z_0|=R$$

once in the counterclockwise direction corresponds to the sum of the fixed vector z_0 and a vector of length R whose angle of inclination θ varies from $\theta=0$ to $\theta=2\pi$.