§2.3. The Exponential Function and its Mapping Properties

1. Operation of exponential function

That chapter will start with the exponential function

$$e^z = e^x e^{iy} = e^x(\cos y + i\sin y) \quad (z = x + iy) \tag{2.3.1}$$

the two factors e^x and e^{iy} being well defined at this time (see Sec.1.6). Note that, definition (2.3.1), which can also be written

$$e^{x+iy} = e^x e^{iy}, \text{ where } e^{iy} = \cos y + i\sin y$$

is suggested by the familiar property $e^{x_1+x_2} = e^{x_1}e^{x_2}$ of the exponential function in calculus.

2. Examples

Example 1. The transformation $w = e^z$ can be written $\rho e^{i\phi} = e^x e^{iy}$, where $z = x + iy$ and $w = \rho e^{i\phi}$. Thus, $\rho = e^x$ and $\phi = y + 2n\pi$, where n is some integer (see Sec.1.8); and this transformation can be expressed in the form

$$\rho = e^x, \phi = y. \tag{2.3.2}$$

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = \exp c_1$ and $\phi = y$ in the w-plane. That image moves counterclockwise around the circle shown in Fig. 2-5 as z moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line.

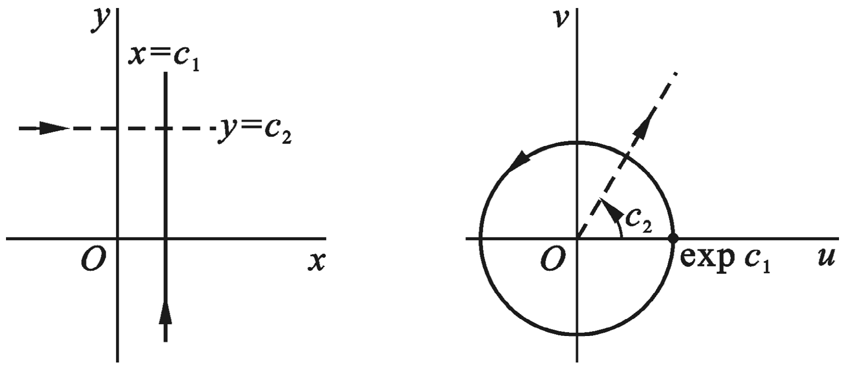

Fig. 2-5

A horizontal line $y = c_2$ is mapped in a one to one manner onto the ray $\phi = c_2$. To see that this is so, we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$ and $\phi = c_2$. Evidently, then, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\phi = c_2$, as indicated in Fig. 2-5.

Vertical and horizontal *line segments* are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in Example 1. This is illustrated in the following example.

Example 2. Let us show that the transformation $w = e^z$ maps the rectangular region

$$\{(x, y) : a \le x \le b, c \le y \le d\}$$

onto the region $\{(\rho, \phi) : e^a \le \rho \le e^b, c \le \phi \le d\}$. The two regions and corresponding parts of their boundaries are indicated in Fig. 2-6. The vertical line segment AD is mapped onto the arc $\rho = e^a$, $c \le \phi \le d$, which is labeled $A'D'$. The images of vertical line segments to the right of AD and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment BC is the arc $\rho = e^b$, $c \le \phi \le d$, labeled $B'C'$. The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \le \phi \le \pi$; and the rectangular region is mapped onto half of a circular ring.

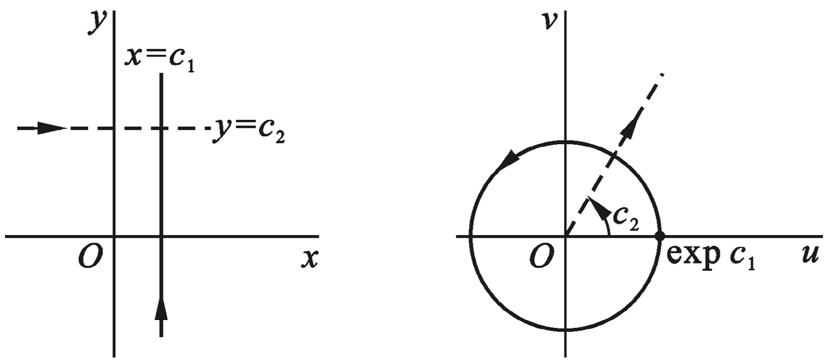

Fig. 2-6

Our final example here uses the images of *horizontal* lines to find the image of a horizontal strip.

Example 3. When $w = e^z$, the image of the infinite strip $0 \le y \le \pi$ is the upper half $v \ge 0$ of the w-plane (Fig. 2-7). This is seen by recalling from Example 1 how a horizontal line $y = c$ is transformed into a ray $\phi = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, and the angles of inclination of the rays increase from $\phi = 0$ to $\phi = \pi$.

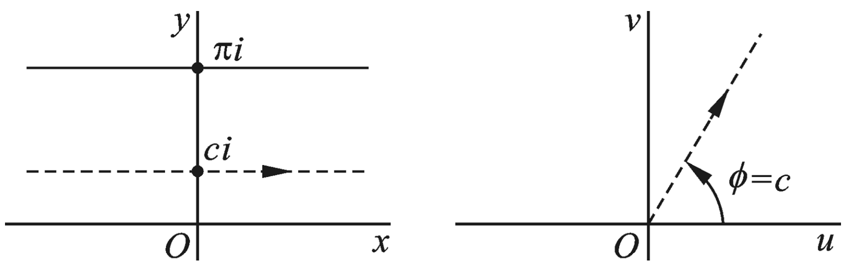

Fig. 2-7