§7.4. Jordan's Lemma

In the evaluation of integrals of the type treated in Sec. 7.3, it is sometimes necessary to use *Jordan's lemma,* which is stated here as a theorem.

Theorem 7.4.1(Jordan). *Suppose that*

(i) *a function* $f(z)$ *is analytic at all points* z *in the upper half plane* $y \geq 0$ *that are exterior to a circle* $|z| = R_0$;

(ii) C_R *denotes a semicircle* $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$, *where* $R > R_0$ (*Fig.* 7-4);

(iii) *there is a constant* M_R *such that* $|f(z)| \leq M_R, \forall z \in C_R$, *and* $\lim_{R\to\infty} M_R = 0$.

Then, for every positive constant a,

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\,dz = 0. \qquad (7.4.1)$$

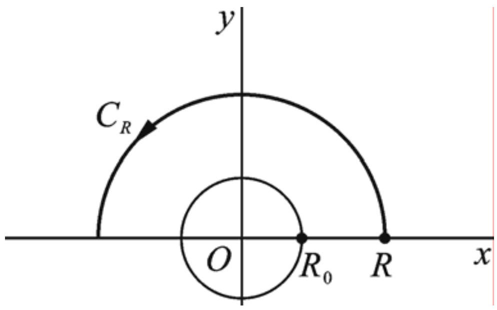


Fig. 7-4

Proof. The proof is based on a result that is known as *Jordan's inequality:*

$$\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R} \quad (R > 0). \qquad (7.4.2)$$

To verify this inequality, we first note from the graphs of the functions $y = \sin\theta$ and $y = 2\theta/\pi$ when $0 \leq \theta \leq \pi/2$ (Fig. 7-5) that $\sin\theta \geq 2\theta/\pi$ for all values of θ in that interval. Consequently, if $R > 0$,

$$e^{-R\sin\theta} \leq e^{-2R\theta/\pi} \quad \text{when} \quad 0 \leq \theta \leq \frac{\pi}{2};$$

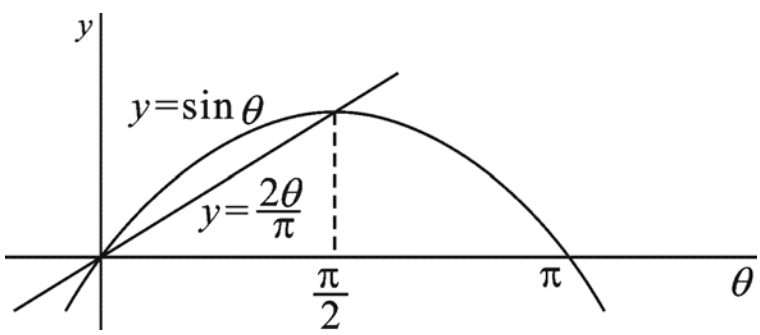
and so

$$\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{\pi}{2R}(1 - e^{-R}).$$

Hence

$$\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta < \frac{\pi}{2R} \quad (R > 0). \qquad (7.4.3)$$

But this is just another form of inequality (7.4.2), since the graph of $y = \sin\theta$ is symmetric with respect to the vertical line $\theta = \pi/2$ on the interval $0 \leq \theta \leq \pi$.



Turning now to the verification of limit (7.4.1), we accept statements $(i)-(iii)$ in the theorem and write

$$\int_{C_R} f(z)e^{iaz}dz = \int_0^{\pi} f(Re^{i\theta})\exp(iaRe^{i\theta})iRe^{i\theta}d\theta .$$

Since

$$|\, f(Re^{i\theta})\,| \le M_R \quad \text{and} \quad |\exp(iaRe^{i\theta})\,| \le e^{-aR\sin\theta}$$

and in view of Jordan's inequality (7.4.2), it follows that

$$\left|\int_{C_R} f(z)e^{iaz}dz\right| \le M_R R\int_0^{\pi} e^{-aR\sin\theta}d\theta$$

$$< \frac{M_R\pi}{a}.$$

Limit (7.4.1) is then evident, since $M_R \to 0$ as $R \to \infty$. The proof is completed.

Example. Let us find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x\sin x dx}{x^2+2x+2}.$$

As usual, the existence of the value in question will be established by our actually finding it.

We write

$$f(z) = \frac{z}{z^2+2z+2} = \frac{z}{(z-z_1)(z-\overline{z_1})},$$

where $z_1 = -1+i$. The point z_1, which lies above the x axis, is a simple pole of the function $f(z)e^{iz}$, with residue

$$B_1 = \frac{z_1 e^{iz_1}}{z_1 - \overline{z_1}}. \tag{7.4.4}$$

Hence, when $R > \sqrt{2}$ and C_R denotes the upper half of the positively oriented circle $|\,z\,| = R$,

$$\int_{-R}^{R} \frac{xe^{ix}dx}{x^2+2x+2} = 2\pi iB_1 - \int_{C_R} f(z)e^{iz}dz\,;$$

and this means that

$$\int_{-R}^{R} \frac{x\sin x dx}{x^2+2x+2} = \mathrm{Im}(2\pi iB_1) - \mathrm{Im}\int_{C_R} f(z)e^{iz}dz\,. \tag{7.4.5}$$

We note that, when z is a point on C_R, $|\, f(z)\,| \le M_R$ where

$$M_R = \frac{R}{(R-\sqrt{2})^2} \to 0(R\to\infty)$$

By Theorem 7.4.1,

$$\left|\mathrm{Im}\int_{C_R} f(z)e^{iz}dz\right| \le \left|\int_{C_R} f(z)e^{iz}dz\right| \to 0(R\to\infty)\,. \tag{7.4.6}$$

Consequently, equation (7.4.5), together with expression (7.4.4) for the residue B_1, tells us that

$$P.V.\int_{-\infty}^{\infty} \frac{x\sin x dx}{x^2+2x+2} = \mathrm{Im}(2\pi iB_1) = \frac{\pi}{e}(\sin 1 + \cos 1)\,. \tag{7.4.7}$$