Chapter Ⅰ

Complex Number Field

In this chapter, we survey the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known. The positive integer number system, integer number system, rational number system and real number system are denoted by $\mathbf{N}, \mathbf{Z}, \mathbf{Q}$ and $\mathbf{R}$, respectively.

§1.1. Sums and Products

1. Definiton of Complex Numbers

A complex number is defined as an ordered pair (x, y) of real numbers x and y.

It is customary to denote a complex number (x, y) by z, so that

$$z = (x, y). \tag{1.1.1}$$

The real numbers x and y are called the *real and imaginary parts* of z, respectively; and we write

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y. \tag{1.1.2}$$

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal whenever they have the same real parts and the same imaginary parts.

2. Operations of Complex Numbers

The *sum* $z_1 + z_2$ and the *product* $z_1 z_2$ of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{1.1.3}$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2). \tag{1.1.4}$$

3. The Relationship of Real Numbers and Complex Numbers

Note that the operations defined by equations (1.1.3) and (1.1.4) become the usual operations of addition and multiplication when restricted to the real numbers:

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0), \quad (x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

The complex number system is, therefore, a natural extension of the real number system.

4. Alternative Representation of Complex Numbers

Any complex number $z = (x, y)$ can be written as $z = (x,0) + (0, y)$, and it is easy to see that $(0,1)(y,0) = (0, y)$. Hence

$$z = (x,0) + (0,1)(y,0);$$

and, if we think of a real number x as the complex number $(x,0)$, that is, we identify a real number x with a corresponding complex number $(x,0)$, and let

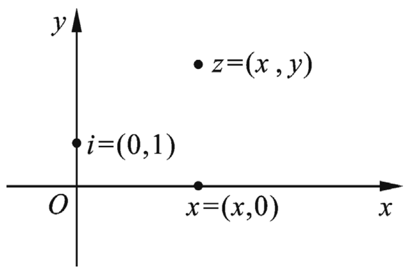

Fig. 1-1

i denote the imaginary number $(0,1)$ (Fig. 1-1)
it is clear that

$$z = x + iy\,, \tag{1.1.5}$$

which is called the rectangular form of the number z. Thus, the complex number system can be written as

$$\mathbf{C} = \{(x, y) : x, y \in \mathbf{R}\} = \{x + iy : x, y \in \mathbf{R}\}.$$

Also, with the convention $z^2 = zz, z^3 = zz^2$, etc., we find that

$$i^2 = (0,1)(0,1) = (-1,0) = -1\,. \tag{1.1.6}$$

Thus, the equation $z^2 + 1 = 0$ has a root $z = i$ in $\mathbf{C}$.

In view of expression (1.1.5), definitions (1.1.3) and (1.1.4) become

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)\,, \tag{1.1.7}$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)\,. \tag{1.1.8}$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left replacing i^2 by -1 when it occurs.