§2.15. Harmonic Functions

1. Definition of an harmonic function

A real-valued function H of two real variables x and y is said to be *harmonic* in a given domain of the xy plane if it has continuous partial derivatives of the first and second order and satisfies the *Laplace's equation*

$$H_{xx}(x,y)+H_{yy}(x,y)=0, \tag{2.15.1}$$

throughout that domain.

Example 1. It is easy to verify that the function $T(x,y)=e^{-y}\sin x$ is harmonic in any domain of the xy plane and, in particular, in the semi-infinite vertical strip $0<x<\pi, y>0$. It also assumes the values on the edges of the strip that are indicated in Fig. 2-15. More precisely, it satisfies all of the conditions

$$T_{xx}(x,y)+T_{yy}(x,y)=0,\ T(0,y)=0,\quad T(\pi,y)=0,$$
$$T(x,0)=\sin x,\quad \lim_{y\to\infty}T(x,y)=0,$$

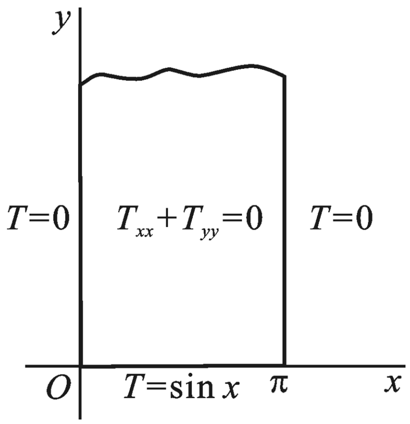

Fig. 2-15

which describe steady temperatures $T(x,y)$ in a thin homogeneous plate in the xy plane that has no heat sources or sinks and is insulated except for the stated conditions along the edges.

Theorem 2.15.1. *If a function $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.*

Example 2. The function

$$f(z)=e^{-y}\sin x-ie^{-y}\cos x$$

is entire, as is shown in Exercise $1(c)$, Sec. 2.14. Hence its real part, which is the temperature function $T(x,y)=e^{-y}\sin x$ in Example 1, must be harmonic in every domain of the xy plane.

Example 3. Since the function $f(z)=z/z^2$ is analytic whenever $z\neq 0$ and since

$$\frac{i}{z^2}=\frac{i}{z^2}\cdot\frac{i\overline{z}^2}{\overline{z}^2}=\frac{i\overline{z}^2}{(z\overline{z})^2}=\frac{i\overline{z}^2}{|z|^4}=\frac{2xy+i(x^2-y^2)}{(x^2+y^2)^2},$$

the two functions

$$u(x,y)=\frac{2xy}{(x^2+y^2)^2}\quad\text{and}\quad v(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}$$

are harmonic throughout any domain in the xy plane that does not contain the origin.

2. Definition of a harmonic conjugate

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equation (2.15.2) throughout D, then v is said to be a *harmonic conjugate* of u.

Theorem 2.15.2. *A function* $f(z)=u(x,y)+iv(x,y)$ *is analytic in a domain* D *if and only if* v *is a harmonic conjugate of* u.

Example 4. Suppose that

$$u(x,y)=x^2-y^2 \text{ and } v(x,y)=2xy.$$

Since these are the real and imaginary components of the entire function $f(z)=z^2$, respectively, we know that v is a harmonic conjugate of u throughout the plane. But u cannot be a harmonic conjugate of v since, as verified in Exercise $2(b)$, Sec. 2.14, the function

$$2xy+i(x^2-y^2)$$

is not analytic anywhere.

Example 5. We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. The function

$$u(x,y)=y^3-3x^2y \tag{2.15.5}$$

is readily seen to be harmonic throughout the entire xy plane. Suppose that v is a harmonic conjugate of u. Then by means of the Cauchy-Riemann equations, we see that

$$u_x=v_y, \quad u_y=-v_x. \tag{2.15.6}$$

The first of these equations tells us that $v_y(x,y)=-6xy$. Holding x fixed and integrating each side here with respect to y, we find that

$$v(x,y)=-3xy^2+\phi(x), \tag{2.15.7}$$

where ϕ is, at present, an arbitrary differentiable function of x. Using the second of equations (2.15.6), we have $3y^2-3x^2=3y^2-\phi'(x)$, and so $\phi'(x)=3x^2$. Thus $\phi(x)=x^3+C$, where C is an arbitrary real number. According to equation (2.15.7), then, the function

$$v(x,y)=-3xy^2+x^3+C \tag{2.15.8}$$

is a harmonic conjugate of $u(x,y)$. The corresponding analytic function is

$$f(z)=(y^3-3x^2y)+i(-3xy^2+x^3+C). \tag{2.15.9}$$

The form $f(z)=i(z^3+C)$ of this function is easily verified and is suggested by noting that when $y=0$, expression (2.15.9) becomes $f(x)=i(x^3+C)$.