§4.7. Primitive Functions

Although the value of a path integral of a function f from a fixed point z_1 to a fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals are from z_1 to z_2 have values that are *independent of path.* (Compare Examples 2 and 3 in Sec. 4.5.) The examples just cited also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. The theorem below is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

In proving the theorem, we shall discover an extension of the fundamental theorem of calculus that simplifies the evaluation of many path integrals. That extension involves the concept of *primitive function*, which is defined as follows.

Definition 4.7.1. For a function f defined on a domain D, if a function F statisfies $F'(z)=f(z)$ for all z in D, then we call F a *primitive function* of f.

Note that a primitive function is necessarily an analytic function. Note, too, that primitive function of a given function f is unique except for an additive complex constant. This is because the derivative of the difference $F(z)-G(z)$ of any two such primitive functions $F(z)$ and $G(z)$ is identically zero on the domain D; and, according to Theorem 2.13.1, an analytic function is constant in a domain D when its derivative is zero throughout D.

Theorem 4.7.1. *Suppose that a function f is continuous on a domain D. Then the following statements are equivalent*:

(i) *f has a primitive function F in D.*

(ii) *The integrals of f along any simple paths lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value.*

(iii) *The integrals of f around any closed simple paths lying entirely in D all have value zero.*

Proof. $(i)\Rightarrow(ii)$: Let us assume that statement (i) is true. Let $z_1, z_2 \in D$ and C be any simple path from z_1 to z_2 lying in D.

When C is a smooth arc, we let $z=z(t)(a\le t\le b)$ be the parametric representation of C, then we know from Exercise 5, Sec. 4.3, that

$$\frac{d}{dt}F[z(t)]=F'[z(t)]z'(t)=f[z(t)]z'(t) \quad (a\le t\le b).$$

Because the fundamental theorem of calculus can be extended so as to apply to complex-value functions of a real variable (Sec. 4.2), it follows that

$$\int_C f(z)dz=\int_a^b f[z(t)]z'(t)dt=F[z(t)]\Big|_a^b=F[z(b)]-F(z(a)].$$

Since $z(b)=z_2$ and $z(a)=z_1$, we may write

$$\int_C f(z)dz=F(z_2)-F(z_1)=F(z)\Big|_{z_1}^{z_2}. \tag{4.7.1}$$

When C consists of a finite number of smooth arcs $C_k\,(k=1,2,\ldots,n)$, let each C_k extend from a point c_k to a point c_{k+1}, where $c_1=z_1, c_{n+1}=z_2$. Then by (4.7.1)

$$\int_C f(z)dz=\sum_{k=1}^{n}\int_{C_k} f(z)dz=\sum_{k=1}^{n}[F(c_{k+1})-F(c_k)]=F(c_{n+1})-F(c_1)=F(z_2)-F(z_1).$$

This shows that the value of the path integral is always, which is evidently independent of the path C as long as C extends from z_1 to z_2 and lies entirely in D.

$(ii)\Rightarrow(iii)$: Let C be any simple closed path lying in D and let z_1 and z_2 denote any two distinct points on C. We divide C into two paths C_1 and C_2, each with initial point z_1 and final point z_2, such that $C=C_1-C_2$ (Fig. 4-14(a)).

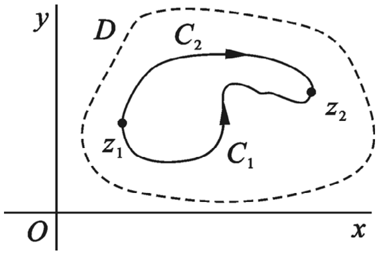

Fig. 4-14(a)

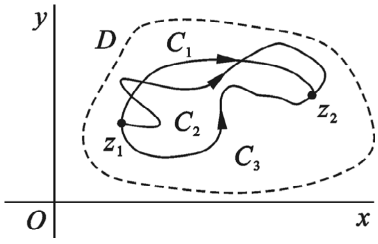

Fig. 4-14(b)

Assuming that statement (ii) is true, one can write

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz, \tag{4.7.2}$$

that is,

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0. \tag{4.7.3}$$

Thus, the integral of f around the closed path $C = C_1 - C_2$ has value zero.

$(iii) \Rightarrow (i)$: We assume that statement (iii) is true. We let C_1 and C_2 denote any two simple paths, lying in D, from a point z_1 to a point z_2 in D. Take a simple path C_3 from z_1 to a point z_2 such that $C_1 - C_3$ and $C_2 - C_3$ are simple paths lying in D (see Fig. 4-14(b)). From (iii), we get that

$$\int_{C_1} f(z)dz + \int_{-C_3} f(z)dz = 0 \quad \text{and} \quad \int_{C_2} f(z)dz + \int_{-C_3} f(z)dz = 0.$$

Thus, equation (4.7.2) holds. Integration is therefore independent of path in D; and we can define a function

$$F(z) = \int_{z_0}^{z} f(s)ds$$

on D where the point z_0 is any fixed point in D. The proof of the theorem is complete once we show that $F'(z) = f(z)$ everywhere in D. Let z be any point in D. Then there is an $r > 0$ such that $z + \Delta z \in D$ when $|\Delta z| < r$. Then, when $0 < |\Delta z| < r$,

$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(s)ds - \int_{z_0}^{z} f(s)ds = \int_{z}^{z+\Delta z} f(s)ds,$$

where the path of integration from z to $z + \Delta z$ may be selected as a line segment (Fig. 4-15). Since

$$\int_{z}^{z+\Delta z} ds = \Delta z$$

(See Exercise 5, Sec. 4.5), we can write

$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)ds;$$

and it follows that

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(s) - f(z)]ds.$$

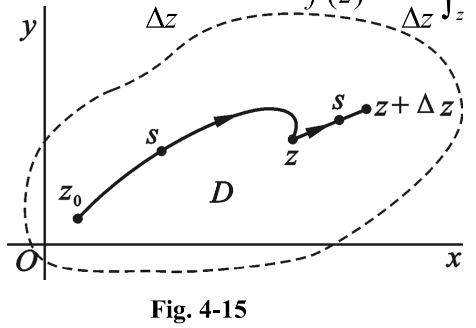

Fig. 4-15

But f is continuous at the point z. Hence, for each positive number ε, there exists a positive number $\delta < r$ such that $|f(s) - f(z)| < \varepsilon$ whenever $|s - z| < \delta$. Consequently, when $0 < |\Delta z| < \delta$, we have from Theorem 4.6.1 that

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| \le \frac{1}{|\Delta z|}\varepsilon |\Delta z| = \varepsilon .$$

This proves that

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z),$$

that is, $F'(z) = f(z)$. Thus, the function F is a primitive function of the function f.

This completes the proof.

From the proof of Theorem 4.7.1 we see that the following result is valid, which gives a formula for computing a path intergral.

Theorem 4.7.2. *Suppose that a function* f *is continuous on a domain* D *and has a primitive function* F *in* D. *If* C *is a simple path from* z_1 *to* z_2 *lying in* D, *then*

$$\int_C f(z)dz = F(z_2) - F(z_1) = F(z)\Big|_{z_1}^{z_2}, \tag{4.7.4}$$

which is called the Newton-Leibnitz formula.

For example, since $(\sin z)' = \cos z (\forall z \in \mathbf{C})$, we have from (4.7.4) that

$\int_A^B \cos z dz = \sin B - \sin A (\forall A, B \in \mathbf{C})$.