§4.11. Extended Cauchy Integral Theorem

The Cauchy integral theorem can be generalized in the following two ways. The first one is the following, called the first extended Cauchy integral theorem.

Theorem 4.11.1(ECIT-1). *If a function* f *is analytic throughout a simply connected domain* D*, then*

$$\int_C f(z)dz = 0 \tag{4.11.1}$$

for every closed path C *lying in* D *that intersects itself at most a finite number of times.*

Proof. *Case* 1: Let C be a simple closed path.

In this case, the function f is analytic at each point interior to and on C; and the Cauchy integral theorem ensures that equation (4.11.1) holds.

Case 2: Let C be a closed path that intersects itself a finite number of times.

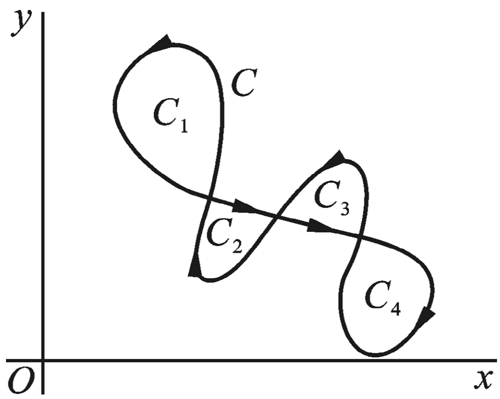

Fig. 4-24

In this case, it consists of a finite number of simple closed paths, illustrated in Fig. 4-24, where the simple closed paths $C_k\,(k = 1,2,3,4)$ make up C.

Since the value of the integral around each C_k is zero, according to the Carchy-Goursat theorem, it follows that

$$\int_C f(z)dz = \sum_{k=1}^{4} \int_{C_k} f(z)dz = 0\,.$$

This completes the proof.

Note that in the case where C is a closed path that has an *infinite* number of self-intersection points, the formula (4.11.1) is also valid. One method that can sometimes be used to show that the theorem still applies in this case is illustrated in Exercise 5 below.

The following corollary follows immediately from Theorem 4.11.1.

Corollary 4.11.1. *A function* f *that is analytic throughout a simply connected domain* D *must have a primitive function everywhere in* D.

Corollary 4.11.1 tells us that entire functions always possess primitive functions.

The Cauchy integral theorem can also be extended in a way that involves integrals along the boundary of a bounded multiply connected domain that is a bounded domaim which is not simple connected. The following theorem is such an extension, called the second extended Cauchy integral theorem.

Theorem 4.11.2(ECIT-2). *Suppose that*

(i) C *is a simple closed path, described in the counterclockwise direction*;

(ii) $C_k (k = 1, 2, \ldots, n)$ *are simple closed paths interior to* C *, all described in the clockwise direction, that are disjoint and whose interiors have no points in common* (Fig.4-25);

(iii) $\overline{D}$ *is a connected region, where*

$$D = \text{ins}(C) \setminus \bigcup_{k=1}^{n} \overline{\text{ins}(C_k)};$$

(iv) A function f *is analytic on*

$$D = \overline{\text{ins}(C)} \setminus \bigcup_{k=1}^{n} \text{ins}(C_k).$$

Then

$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0. \tag{4.11.2}$$

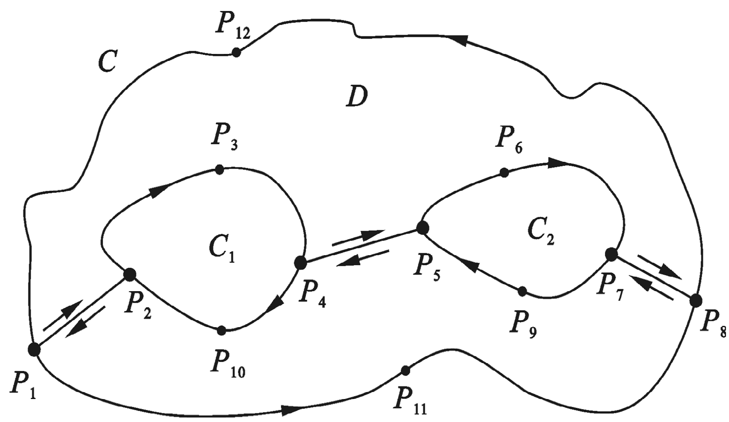

Fig. 4-25

Note that, in equation (4.11.2), the direction of each path of integration is such that the multiply connected domain lies to the *left* of that path.

Proof. To specify, let us give the proof for the case where $n = 2$. Choose points

$$P_i \in C + C_1 + C_2 (i = 1, 2, \ldots, 12)$$

as in Fig. 4-25. Since the region $\overline{D}$ is connected, we can find polygonal lines, say line segments, $P_1P_2, P_4P_5, P_7P_8 \subset \overline{D}$ such that

$$\Gamma_1 = P_1P_2 + P_2P_3P_4 + P_4P_5 + P_5P_6P_7 + P_7P_8 + P_8P_{12}P_1$$

and

$$\Gamma_2 = P_1P_{11}P_8 + P_8P_7 + P_7P_9P_5 + P_5P_4 + P_4P_{10}P_2 + P_2P_1$$

are all simple closed paths.

Since the function f is analytic in both $\overline{\text{ins}(\Gamma_1)}$ and $\overline{\text{ins}(\Gamma_2)}$, it follows from the Cauchy integral theorem that

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = 0 + 0 = 0.$$

On the other hand, because that

$$\int_{P_1P_2} f(z)dz + \int_{P_2P_1} f(z)dz = 0,$$

$$\int_{P_4P_5} f(z)dz + \int_{P_5P_4} f(z)dz = 0,$$

$$\int_{P_7P_8} f(z)dz + \int_{P_8P_7} f(z)dz = 0,$$

$$\int_{P_2P_3P_4} f(z)dz + \int_{P_4P_{10}P_2} f(z)dz = \int_{C_1} f(z)dz,$$

$$\int_{P_5P_6P_7} f(z)dz + \int_{P_7P_9P_5} f(z)dz = \int_{C_2} f(z)dz,$$

$$\int_{P_8P_{12}P_1} f(z)dz + \int_{P_1P_{11}P_8} f(z)dz = \int_{C} f(z)dz,$$

we get that

$$\int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C} f(z)dz = 0.$$

This completes the proof.

The following corollary is an especially important consequence of Theorem 4.11.2.

Corollary 4.11.2. *Let* C_1 *and* C_2 *denote positively oriented simple closed paths, where* C_2 *is interior to* C_1 (Fig. 4-26). *If a function* f *is analytic in the closed region consisting of those paths and all points between them, then*

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz. \qquad (4.11.3)$$

Proof. We use Theorem 4.11.2 to write

$$\int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = 0;$$

and we note that this is just a different form of equation (4.11.3).

Corollary 4.11.2 is known as the *principle of deformation of paths* since it tells us that if C_1 is continuously deformed onto C_2, always passing through points at which f is analytic, then the value of the integral of f over C_1 never changes.

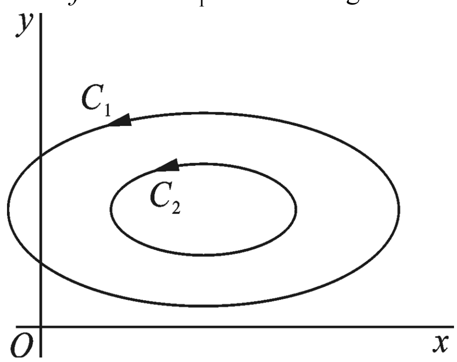

Fig. 4-26

Example. When C is any positively oriented simple closed path surrounding the origin, Corollary 4.11.2 can be used to show that

$$\int_C \frac{dz}{z} = 2\pi i.$$

To do this, we need only construct a positively oriented circle C_0 with center at the origin and radius so small that C_0 lies entirely inside C (Fig. 4-27). Since [Exercise $10(a)$, Sec. 4.5]

$$\int_{C_0} \frac{dz}{z} = 2\pi i$$

and since $1/z$ is analytic everywhere except at $z = 0$, the desired result follows.

Note that the radius of C_0 could equally well have been so large that C lies entirely inside C_0.

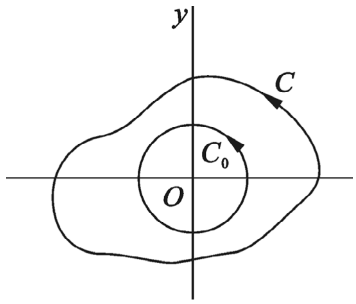

Fig. 4-27