§3.3. Branches and Derivatives of Logarithms

1. Properties of the branche $\mathrm{L}_\alpha(z)$

Put $D_\alpha = \{re^{i\theta} : r > 0, \alpha < \theta < \alpha + 2\pi\}$, a function $L_\alpha : D_\alpha \to \mathbf{C}$ defined by

$$\mathrm{L}_\alpha(z) = \ln r + i\theta \ (z = re^{i\theta}, r > 0, \alpha < \theta < \alpha + 2\pi). \qquad (3.3.2)$$

From this definition, we can prove that the function $L_\alpha(z)$ has the following properties.

(1) $e^{L_\alpha(z)} = z(\forall z \in D_\alpha)$;

(2) $f_n(z) = L_{(2n-1)\pi}(z), \forall n \in \mathbf{Z}$, whenever $-\pi < \arg z < \pi$;

(3) $\mathrm{Log} z = \{L_\alpha(z) : \alpha \in \mathbf{R}\}(\forall z \in \mathbf{C} \setminus \{0\})$;

(4) $L_\alpha(D_\alpha) = \{(u, v) : u \in \mathbf{R}, \alpha < v < \alpha + 2\pi\}$;

(5) $\frac{d}{dz}\mathrm{L}_\alpha(z) = \frac{1}{z} \ (z = re^{i\theta}, r > 0, \alpha < \theta < \alpha + 2\pi)$.

2. Definition of a branch of a multi-valued function

A *branch* of a multi-valued function F defined on D is any single-valued function $f : E \to \mathbf{C}$ such that

(1) f is analytic on the domain E;

(2) $E \subset D$; and

(3) $\forall z \in E$, $f(z) \in F(z)$.