§8.9. Mappings of the Upper Half Plane

Let us determine all fractional linear transformations that map the upper half plane $\operatorname{Im} z > 0$ onto the open disk $|w| < 1$ and the boundary $\operatorname{Im} z = 0$ onto the boundary $|w| = 1$ (Fig. 8-8).

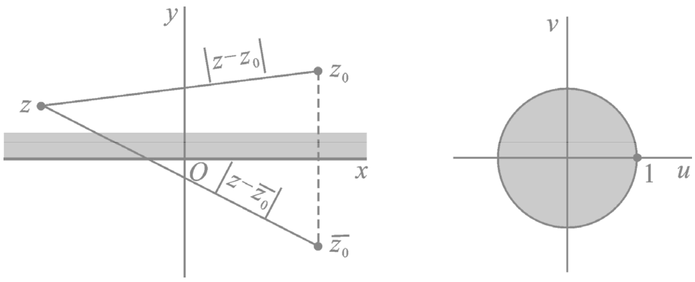

Fig. 8-8

Keeping in mind that points on the line $\operatorname{Im} z = 0$ are to be transformed into points on the circle $|w| = 1$, we start by selecting the points $z = 0, z = 1$, and $z = \infty$ on the line and determining conditions on a fractional linear transformation

$$w = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0) \tag{8.9.1}$$

which are necessary in order for the images of those points to have unit modulus.

We note from equation (8.9.1) that if $|w| = 1$ when $z = 0$, then $|b/d| = 1$; that is,

$$|b| = |d| \neq 0 . \tag{8.9.2}$$

Now, according to Sec. 8.7, the image w of the point $z = \infty$ is the finite point $w = a/c$, we see that $|a/c| = 1$, i.e.

$$|a| = |c| \neq 0 . \tag{8.9.3}$$

The fact that a and c are nonzero enables us to rewrite equation (8.9.1) as

$$w = \frac{a}{c} \cdot \frac{z + (b/a)}{z + (d/c)} . \tag{8.9.4}$$

Then, since $|a/c| = 1$ and

$$\left|\frac{b}{z}\right| = \left|\frac{d}{c}\right| \neq 0 ,$$

according to relations (8.9.2) and (8.9.3), equation (8.9.4) can be put in the form

$$w = e^{i\alpha} \frac{z - z_0}{z - z_1} \quad (|z_1| = |z_0| \neq 0) , \tag{8.9.5}$$

where α is a real constant and z_0 and z_1 are (nonzero) complex constants.

Next, we impose on transformation (8.9.5) the condition that $|w| = 1$ when $z = 1$. This tells us that $|1 - z_1| = |1 - z_0|$, thus

$$(1 - z_1)(1 - \overline{z_1}) = (1 - z_0)(1 - \overline{z_0}) .$$

But $z_1 \overline{z_1} = z_0 \overline{z_0}$ since $|z_1| = |z_0|$, and the above relation reduces to

$$z_1 + \overline{z_1} = z_0 + \overline{z_0} ,$$

again since $|z_1| = |z_0|$. If $z_1 = z_0$, then transformation (8.9.5) becomes the constant function $w = \exp(i\alpha)$; hence $z_1 = \overline{z_0}$.

Transformation (8.9.5), with $z_1 = \overline{z_0}$, maps the point z_0 onto the origin $w = 0$; and, since points interior to the circle $|w| = 1$ are to be the images of points above the real axis in the z plane, we may conclude that $\operatorname{Im} z_0 > 0$. Therefore, *any linear fractional transformation having the desired property must be of the form*

$$w = e^{i\alpha} \frac{z - z_0}{z - \overline{z_0}} \quad (\operatorname{Im} z_0 > 0), \tag{8.9.6}$$

where α *is real.*

It remains to show that, conversely, any fractional linear transformation of the form (8.9.6) has the desired mapping property. This is easily done by taking absolute values of each side of equation (8.9.6) and interpreting the resulting equation,

$$|w| = \left| \frac{z - z_0}{z - \overline{z_0}} \right|,$$

geometrically. If a point z lies above the real axis, both it and the point z_0 lie on the same side of that axis, which is the perpendicular bisector of the line segment joining z_0 and $\overline{z_0}$. It follows that the distance $|z - z_0|$ is less than the distance $|z - \overline{z_0}|$ (Fig.8-8); that is, $|w| < 1$. Likewise, if z lies below the real axis, the distance $|z - z_0|$ is greater than the distance $|z - \overline{z_0}|$; and so $|w| > 1$. Finally, if z is on the real axis, $|w| = 1$ because then $|z - z_0| = |z - \overline{z_0}|$. Since any fractional linear transformation is a one to one mapping of the extended z plane onto the extended w plane, this shows that *transformation* (8.9.6) *maps the half plane* $\operatorname{Im} z > 0$ *onto the disk* $|w| < 1$ *and the boundary of the half plane onto the boundary of the disk.*

As a summary, we have

Theorem 8.9.1. *A linear fractional transformation* maps *the upper half plane* $\operatorname{Im} z > 0$ *onto the open disk* $|w| < 1$ *and the boundary* $\operatorname{Im} z = 0$ *onto the boundary* $|w| = 1$ *if and only if it has of the form*

$$w = e^{i\alpha} \frac{z - z_0}{z - \overline{z_0}} \quad (\operatorname{Im} z_0 > 0, \operatorname{Im} \alpha = 0). \tag{8.9.7}$$

Our first example here illustrates the use of the result in italics just above.

Example 1. The transformation $w = \dfrac{i - z}{i + z}$ in Examples 1 in Secs. 8.7 and 8.8 can be written $w = e^{i\pi} \dfrac{z - i}{z - \overline{i}}$. Hence, it has the mapping property described in italics.

Images of the upper half plane $\operatorname{Im} z \geq 0$ under other types of linear fractional transformations are often fairly easy to determine by examining the particular transformation in question.

Example 2. By writing $z = x + iy$ and $w = u + iv$, show that the transformation $w = \dfrac{z - 1}{z + 1}$ maps the half plane $y > 0$ onto the half plane $v > 0$ and the x axis onto the u axis.

Proof. We first note that when the number z is real, so is the number w. Consequently, since the image of the real axis $y = 0$ is either a circle or a line, it must be the real axis $v = 0$. Furthermore, for any point w in the finite w plane,

$$v = \operatorname{Im} w = \operatorname{Im} \frac{(z - 1)(\overline{z} + 1)}{(z + 1)(\overline{z + 1})} = \frac{2y}{|z + 1|^2} \quad (z \neq -1).$$

The numbers y and v thus have the same sign, and this means that points above the x axis correspond to points above the u axis and points below the x axis correspond to points below the u axis. Finally, since point on the x axis correspond to points on the u axis and since a fractional linear transformation is a one to one mapping of the extended plane onto the extended plane (Sec. 8.7), the stated mapping property of the transformation is established.

Our final example involves a composite function and uses the map discussed in Example 2.

Example 3. The transformation

$$w = \log\frac{z-1}{z+1}, \tag{8.9.8}$$

where the principal branch of the logarithmic function is used, is a composition of the functions

$$Z = \frac{z-1}{z+1} \quad \text{and} \quad w = \log Z. \tag{8.9.9}$$

We know from Example 2 that the first of transformations (8.9.9) maps the upper half plane $y > 0$ onto the upper half plane $Y > 0$, where $z = x + iy$ and $Z = X + iY$. Furthermore, it is easy to see from Fig. 8-9 that the second of mapping (8.9.9) maps the half plane $Y > 0$ onto the strip $0 < v < \pi$, where $w = u + iv$. More precisely, by writing $Z = R\exp(i\theta)$ and

$$\log Z = \ln R + i\theta \qquad (R > 0, -\pi < \theta < \pi),$$

we see that as a point $Z = R\exp(i\theta_0)(0 < \theta_0 < \pi)$ moves outward from the origin along the ray $\theta = \theta_0$, its image is the point whose rectangular coordinates in the w plane are $(\ln R, \theta_0)$. That image evidently moves to the right along the entire length of the horizontal line $v = \theta_0$. Since these lines fill the strip $0 < v < \pi$ as the choice of θ_0 varies between $\theta_0 = 0$ to $\theta_0 = \pi$, the mapping of the half plane $Y > 0$ onto the strip is, in fact, one to one.

This shows that the composition (8.9.8) of the mappings (8.9.9) transforms the plane $y > 0$ onto the strip $0 < v < \pi$.

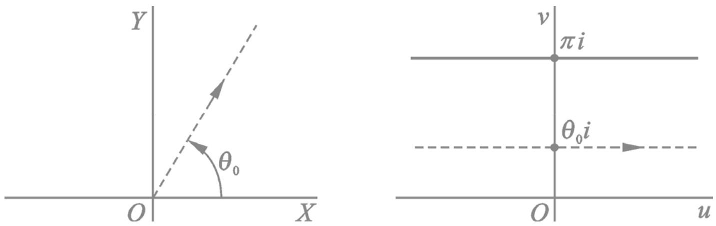

Fig. 8-9

As an application of Theorem 8.9.1, let us find a fractional linera transformation that maps the unit disk $|z| < 1$ in the z-plane onto the the unit disk $|w| < 1$ in the w-plane and a point z_0 in the disk onto the origen of the w-plane, as well as the circle $|z| = 1$ onto the circle $|w| = 1$.

To do this, take a point ω_0 in the ω-plane with $\operatorname{Im}\omega_0 > 0$, then from Theorem 8.9.1 we know that the mapping

$$z = F_1(\omega) = \frac{\omega - \omega_0}{\omega - \overline{\omega_0}} \tag{8.9.10}$$

maps the upper half plane $\operatorname{Im}\omega > 0$ in the ω-plane onto the unit disk $|z| < 1$ in the z-plane, and $\operatorname{Im}\omega = 0$ onto $|z| = 1$. Put $w_1 = F_1^{-1}(z_0)$, then from Theorem 8.9.1, the mapping

$$w = F_2(\omega) = e^{i\alpha}\frac{\omega - \omega_1}{\omega + \overline{\omega_1}} \quad (\operatorname{Im}\alpha = 0) \tag{8.9.11}$$

maps the upper half plane $\operatorname{Im}\omega > 0$ in the ω-plane disk onto the unit disk $|w| < 1$ in the w-plane, and $\operatorname{Im}\omega = 0$ onto $|w| = 1$. See Fig. 8-10.

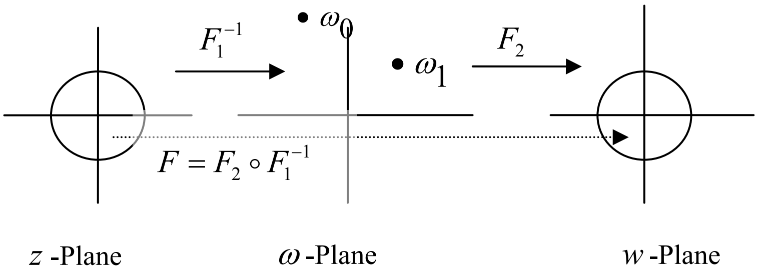

Fig. 8-10

Then the mapping $F = F_2 \circ F_1^{-1}$ satisfies the desired conditions. From (8.9.10), we get $\omega = \dfrac{\overline{\omega_0} z - \omega_0}{z - 1}$, and then by (8.9.11) we know that

$$w = F(z) = F_2(F_1^{-1}(z)) = e^{i\alpha}\lambda\frac{z - z_0}{1 - \overline{z_0} z}, \tag{8.9.12}$$

where $\lambda = \dfrac{\overline{\omega_1} - \omega_0}{\overline{\omega_0} - \omega_1}$. Clearly $|\lambda| = 1$. Thus we may write $e^{i\alpha}\lambda = e^{i\theta}$ where θ is real. Hence, (8.9.12) becomes

$$w = F(z) = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0} z}. \tag{8.9.13}$$