§4.12. Cauchy Integral Formula

Another fundamental result will now be established.

Theorem 4.12.1(Cauchy). *Let f be analytic everywhere inside and on a simple closed path C, taken in the positive sense. If z_0 is any point interior to C, then*

$$f(z_0)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz. \text{ (CIF)} \tag{4.12.1}$$

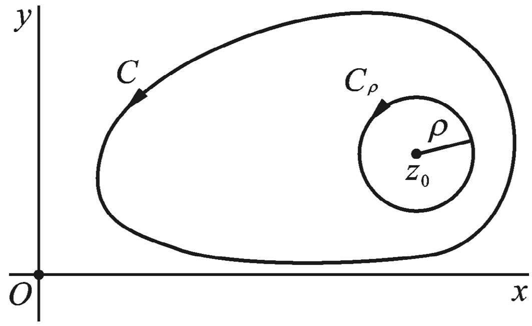

Fig. 4-31

Proof. We begin the proof of the theorem by letting C_ρ denote a positively oriented circle $|z-z_0|=\rho$, where ρ is small enough that C_ρ is interior to C (see Fig. 4-31). Since the function $f(z)/(z-z_0)$ is analytic between and on the paths C and C_ρ, it follows from the principle of deformation of paths (Corollary 4.11.2, Sec.4.11) that

$$\int_C \frac{f(z)}{z-z_0}dz=\int_{C_\rho}\frac{f(z)}{z-z_0}dz.$$

This enables us to write

$$\int_C \frac{f(z)}{z-z_0}dz - f(z_0)\int_{C_\rho}\frac{1}{z-z_0}dz=\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz. \tag{4.12.2}$$

But [see Exercise $10(a)$, Sec. 4.5] $\int_{C_\rho}\frac{1}{z-z_0}dz=2\pi i$; and so equation (4.12.2) becomes

$$\int_C \frac{f(z)dz}{z-z_0}-2\pi i f(z_0)=\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz. \tag{4.12.3}$$

Now the fact that f is analytic and so continuous at z_0 ensures that $\forall\varepsilon$, $\exists\delta>0$ such that

$$|f(z)-f(z_0)|<\varepsilon \text{ whenever } |z-z_0|<\delta. \tag{4.12.4}$$

Then for all $0<\rho<\delta$, we have that $|z-z_0|=\rho<\delta$ when z is on C_ρ, it follows that the first of inequalities (4.12.4) holds. Thus, inequality (4.6.1) tells us that

$$\left|\int_{C_\rho}\frac{f(z)-f(z_0)}{z-z_0}dz\right|\le\frac{\varepsilon}{\rho}2\pi\rho=2\pi\varepsilon, \text{ whenever } 0<\rho<\delta.$$

In view of equation (4.12.3), then,

$$\left|\int_C \frac{f(z)dz}{z-z_0}-2\pi i f(z_0)\right|\le 2\pi\varepsilon.$$

Since the left-hand side of this inequality is a nonnegative constant that is less than an arbitrarily small positive number, it must equal to zero. Hence equation (4.12.1) is valid, and the theorem is proved.

Formula (4.12.1) is called the *Cauchy integral formula* (CIF). It tells us that if a function f is analytic within and on a simple closed path C, then the values of f interior to C are completely determined by the values of f on C.

When the Cauchy integral formula is written

$$\int_C \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0), \tag{4.12.5}$$

it can be used to evaluate the integral on the left hand side of (4.12.5).

Example 1. Find the integral

$$\int_C \frac{zdz}{(9 - z^2)(z + i)},$$

where C is the positively oriented circle $|z| = 2$.

Solution. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, formula (4.12.5) tells us that

$$\int_C \frac{zdz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)} dz = 2\pi i \frac{z}{9 - z^2}\bigg|_{z=-i} = 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}.$$

By using a similar method, one can prove the following result, called the *extended Cauchy integral formula* (ECIF).

Theorem 4.12.2. *Suppose that*

(i) C *is a simple closed path, described in the counterclockwise direction*;

(ii) $C_k (k = 1, 2, \ldots, n)$ *are simple closed paths interior to* C, *all described in the clockwise direction, that are disjoint and whose interiors have no points in common* (Fig. 4-25);

(iii) $\overline{D}$ *is a connected region, where* $D = \mathrm{ins}(C) \setminus \bigcup_{k=1}^{n} \overline{\mathrm{ins}(C_k)}$;

(iv) *A function* f *is analytic on* $\overline{D} = \overline{\mathrm{ins}(C)} \setminus \bigcup_{k=1}^{n} \mathrm{ins}(C_k)$,

then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz + \frac{1}{2\pi i} \sum_{k=1}^{n} \int_{C_k} \frac{f(z)}{z - z_0} dz, \quad \forall z_0 \in D. \tag{4.12.6}$$