§6.9. Zeros and Poles

The following theorem shows how zeros of order m can create poles of order m.

Theorem 6.9.1. *Suppose that*

(i) *Two functions* p *and* q *are analytic at a point* z_0*;*

(ii) $p(z_0) \neq 0$ *and* q *has a zero of order* m *at* z_0.

Then the quotient $p(z)/q(z)$ *has a pole of order* m *at* z_0.

Proof. Since q has a zero of order m at z_0, we know from Theorem 6.7.2 in Sec. 6.7 that there is a deleted neighborhood of z_0 in which $q(z) \neq 0$; and so z_0 is an isolated singular point of the quotient $p(z)/q(z)$. Theorem 6.7.1 in Sec. 6.7 tells us, moreover, that

$$q(z) = (z - z_0)^m g(z),$$

on some neighborhood of z_0, where g is analytic and nonzero at z_0; and this enables us to write

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z - z_0)^m}, \tag{6.9.1}$$

on some deleted neighborhood of z_0. Since $p(z)/g(z)$ is analytic and nonzero at z_0, it now follows from Theorem 6.5.1 that z_0 is a pole of order m of $p(z)/q(z)$. This completes the proof.

Example 1. The two functions

$$p(z) = 1 \quad \text{and} \quad q(z) = z(e^z - 1)$$

are entire; and we know from the example in Sec. 6.7 that q has a zero of order $m = 2$ at the point $z_0 = 0$. Hence it follows from Theorem 6.9.1 that the quotient

$$\frac{p(z)}{q(z)} = \frac{1}{z(e^z - 1)}$$

has a pole of order 2 at that point. This was demonstrated in another way in Example 5, Sec. 6.6.

Theorem 6.9.1 leads us to another method for identifying *simple* poles and finding the corresponding residues. This method is sometimes easier to use that the one in Sec. 6.5.

Theorem 6.9.2. *Let two functions* p *and* q *be analytic at a point* z_0*. If*

$$p(z_0) \neq 0, \quad q(z_0) = 0, \text{ and } q'(z_0) \neq 0,$$

then z_0 *is a simple pole of the quotient* $p(z)/q(z)$ *and*

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{6.9.2}$$

Proof. To show this, we observe that, because of the conditions on q, the point z_0 is a zero of order $m = 1$ of that function. According to Theorem 6.7.1 in Sec. 6.7, then,

$$q(z) = (z - z_0) g(z) \tag{6.9.3}$$

where $g(z)$ is analytic and nonzero at z_0. Furthermore, Theorem 6.9.1 tells us that z_0 is a simple pole of $p(z)/q(z)$; and equation (6.9.1) becomes

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{z - z_0}.$$

Now $p(z)/g(z)$ is analytic and nonzero at z_0, and it follows from Theorem 6.5.1 that

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)}. \tag{6.9.4}$$

But $g(z_0) = q'(z_0)$, as is seen by differentiating each side of equation (6.9.3) and setting $z = z_0$. Expression (6.9.4) thus takes the form (6.9.2). The proof is complete.

Example 2. Consider the function

$$f(z) = \cot z = \frac{\cos z}{\sin z},$$

which is a quotient of the entire functions $p(z) = \cos z$ and $q(z) = \sin z$. The singularities of that quotient occur at the points

$$z = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Since

$$p(n\pi) = (-1)^n \neq 0, \quad q(n\pi) = 0, \text{ and } q'(n\pi) = (-1)^n \neq 0,$$

each singular point $z = n\pi$ of f is a simple pole, with residue

$$B_n = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1.$$

Example 3. The residue of the function

$$f(z) = \frac{\tanh z}{z^2} = \frac{\sinh z}{z^2 \cosh z}$$

at the zero $z = \pi i/2$ of $\cosh z$ (see Sec. 3.7) is readily found by writing

$$p(z) = \sinh z \quad \text{and} \quad q(z) = z^2 \cosh z.$$

Since

$$p\left(\frac{\pi i}{2}\right) = \sinh\left(\frac{\pi i}{2}\right) = i \sin \frac{\pi}{2} = i \neq 0$$

and

$$q\left(\frac{\pi i}{2}\right) = 0, \quad q'\left(\frac{\pi i}{2}\right) = \left(\frac{\pi i}{2}\right)^2 \sinh\left(\frac{\pi i}{2}\right) = -\frac{\pi^2}{4} i \neq 0,$$

we find that $z = \pi i/2$ is a simple pole of f and that the residue there is

$$B = \frac{p(\pi i/2)}{q'(\pi i/2)} = -\frac{4}{\pi^2}.$$

Example 4. One can find the residue of the function

$$f(z) = \frac{z}{z^4 + 4}$$

at the isolated singular point

$$z_0 = \sqrt{2} e^{i\pi/4} = 1 + i$$

by writing $p(z) = z$ and $q(z) = z^4 + 4$. Since

$$p(z_0) = z_0 \neq 0, \quad q(z_0) = 0, \text{ and } q'(z_0) = 4z_0^3 \neq 0,$$

f has a simple pole at z_0. The corresponding residue is the number

$$B_0 = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = -\frac{i}{8}.$$

Although this residue could also be found by the method of Sec. 6.4, the computation would be somewhat more involved.

There are formulas similar to formula (6.9.2) for residues at poles of higher order, but they are lengthier and, in general, not practical.