§8.7. Fractional Linear Transformations

The transformation

$$w = \frac{az+b}{cz+d} \qquad (ad-bc \neq 0), \tag{8.7.1}$$

where a, b, c, and d are complex constants, is called a *fractional linear transformation*(FLT), or Mobius transformation. Observe that equation (8.7.1) can be written in the form

$$Azw + Bz + Cw + D = 0 \qquad (AD - BC \neq 0); \tag{8.7.2}$$

and, conversely, any equation of type (8.7.2) can be put in he form (8.7.1).

When $c = 0$, the condition $ad - bc \neq 0$ with equation (8.7.1) becomes $ad \neq 0$; and we see that the transformation reduces to a nonconstant linear function. When $c \neq 0$, equation (8.7.1) can be written

$$w = \frac{a}{c} + \frac{bc-ad}{c} \cdot \frac{1}{cz+d} \qquad (ad-bc \neq 0). \tag{8.7.3}$$

So, once again, the condition $ad - bc \neq 0$ ensures that we do not have a constant function. The transformation $w = 1/z$ is evidently a special case of transformation (8.7.1) when $c \neq 0$.

Equation (8.7.3) reveals that when $c \neq 0$, a fractional linear transformation is a composition of the mappings.

$$Z = cz + d, \quad W = \frac{1}{Z}, \quad w = \frac{a}{c} + \frac{bc-ad}{c} W \qquad (ad-bc \neq 0).$$

It thus follows from Theorem 8.6.1 that, regardless of whether c is zero or nonzero, the following conclusion holds.

Theorem 8.7.1. *Any linear fractional transformation transforms circles and lines onto circles and lines.*

For a fractional transforntion (8.7.1), we compute that

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

for all $z \neq -d/c$ since $ad - bc \neq 0$. This shows that every fractional linear transformation (8.8.2) is a conformal mappins throughout its domain of definition.

Solving equation (8.7.1) for z, we find that

$$z = \frac{-dw+b}{cw-a} \quad (ad-bc \neq 0). \tag{8.7.4}$$

When a given point w is the image of some point z under transformation (8.7.1), the point z is retrieved by means of equation (8.7.4). If $c = 0$, so that a and d are both nonzero, each point in the w plane is evidently the image of one and only one point in the z plane. The same is true if $c \neq 0$, except when $w = a/c$ since the denominator in equation (8.7.4) vanishes if w has that value. We can, however, enlarge the domain of definition of transformation (8.7.1) in order to define a fractional linear transformation T on the *extended* z plane such that the point $w = a/c$ is the image of $z = \infty$ when $c \neq 0$. We first write

$$T(z) = \frac{az+b}{cz+d} \qquad (ad-bc \neq 0). \tag{8.7.5}$$

We then write

$$T(\infty) = \infty \ \text{ if } \ c = 0$$

and

$$T(\infty) = \frac{a}{c} \ \text{ and } \ T\left(-\frac{d}{c}\right) = \infty \ \text{ if } \ c \neq 0.$$

This makes T continuous on the extended z plane. It also agrees with the way in which we enlarged the domain of definition of the transformation $w = 1/z$ in Sec. 8.5.

When its domain of definition is enlarged in this way, the fractional linear transformation (8.7.5) is a *one to one* mapping of the extended z plane *onto* the extended w plane. That is,

$T(z_1) \neq T(z_2)$ whenever $z_1 \neq z_2$; and, for each point w in the second plane, there is a point z in the first one such that $T(z) = w$. Hence, associated with the transformation T, there is an inverse transformation T^{-1}, which is defined on the extended w plane as follows:

$$T^{-1}(w) = z \quad \text{if and only if } T(z) = w.$$

From equation (8.7.4), we see that

$$T^{-1}(w) = \frac{-dw + b}{cw - a} \quad (ad - bc \neq 0). \tag{8.7.6}$$

Evidently, T^{-1} is itself a fractional linear transformation, where

$$T^{-1}(\infty) = \infty \quad \text{if } c = 0$$

and

$$T^{-1}\left(\frac{a}{c}\right) = \infty \quad \text{and} \quad T^{-1}(\infty) = -\frac{d}{c} \quad \text{if } c \neq 0.$$

If T and S are two fractional linear transformations, then so is the composition $S \circ T$. This can be verified by combining expressions of the type (8.7.5). Note that, in particular, $T^{-1}[T(z)] = z$ for each point z in the extended plane.

There is always a fractional linear transformation that maps three given distinct points z_1, z_2, and z_3 onto three specified distinct points w_1, w_2, and w_3, respectively. This will be proded next section. We illustrate here a direct approach to finding the desired transformation.

Example 1. Let us find the special case of transformation (8.7.1) that maps the points

$$z_1 = -1, \quad z_2 = 0, \text{ and } z_3 = 1$$

onto the points

$$w_1 = -i, \quad w_2 = 1, \text{ and } w_3 = i.$$

Since 1 is the image of 0, expression (8.7.1) tells us that $1 = b/d$, i.e., $d = b$. Thus

$$w = \frac{az + b}{cz + b} \quad (b(a - c) \neq 0). \tag{8.7.7}$$

Then, since -1 and 1 are transformed into $-i$ and i, respectively, it follows that

$$ic - ib = -a + b \quad \text{and} \quad ic + ib = a + b.$$

Adding corresponding sides of these equations, we find that $c = -ib$; and subtraction reveals that $a = ib$. Consequently,

$$w = \frac{ibz + b}{-ibz + b} = \frac{b(iz + 1)}{b(-iz + 1)}.$$

Since b is arbitrary and nonzero here, we may assign it the value unity (or cancel it out) and write

$$w = \frac{iz + 1}{-iz + 1} \cdot \frac{i}{i} = \frac{i - z}{i + z}.$$

Example 2. Suppose that the points

$$z_1 = 1, \quad z_2 = 0, \text{ and } z_3 = -1$$

are mapped onto

$$w_1 = i, \quad w_2 = \infty, \text{ and } w_3 = 1$$

under (8.7.1). Find the coefficients a, b, c and d of (8.7.1).

Solution. Since $w_2 = \infty$ corresponds to $z_2 = 0$, we require that $d = 0$ in expression (8.7.1) ; and so

$$w = \frac{az + b}{cz} \quad (bc \neq 0). \tag{8.7.8}$$

Because 1 is to be mapped onto i and -1 onto 1, we have the relations

$$ic = a + b \quad \text{and} \quad -c = -a + b;$$

and it follows that

$$b = \frac{i-1}{2}c, \quad a = \frac{i+1}{2}c.$$

Finally, then, if we take $c = 2$, then we obtain the desired fractional linera transformation:

$$w = \frac{(i+1)z + (i-1)}{2z}.$$