§6.8. Uniquely Determined Analytic Functions

We conclude this chapter whit two sections dealing with how the values of an analytic function on a domain D are affected by its values in a subdomain or on a line segment lying in D. While these sections are of considerable theoretical interest, they are not central to our development of analytic function in later chapters. The reader may pass directly to Chap. 3 at this time and refer back when necessary.

Lemma 6.8.1. *Suppose that* f *is analytic on a domain* D *and that* $f(z)=0$ *on a domain or a line segment* K *contained in* D*. Then* $f(z)\equiv 0$ *in* D*.*

Proof. Let z_0 be any point of K and P be any other point in D. Since D is a connected open set (Sec. 1.10), there is a polygonal line L, consisting of a finite number of line segments joined end to end and lying entirely in D, that extends from z_0 to P. We let d be the shortest distance from points on L to the boundary of D, unless D is the entire plane; in that case, d may be positive number. We then form a finite sequence of points

$$z_0, z_1, z_2, \ldots, z_{n-1}, z_n$$

along L, where the point z_n coincides with P (Fig. 6-9) and where each points is sufficiently close to the adjacent ones so that $|z_k - z_{k-1}| < d \ (k=1,2,\ldots,n)$.

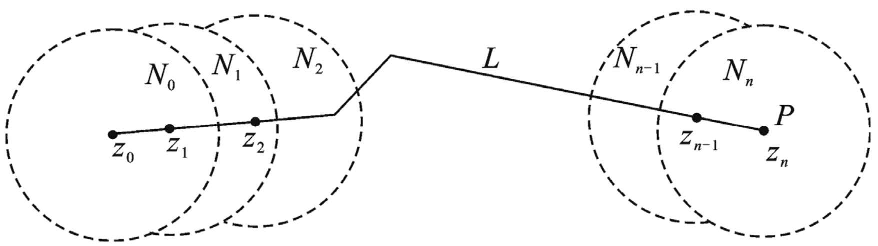

Fig. 6-9

Finally, we construct a finite sequence of neighborhoods

$$N_0, N_1, N_2, \ldots, N_{n-1}, N_n,$$

where each neighborhood N_k is centered at z_k and has radius d. Note that these neighborhoods are all contained in D and that the center z_k of any neighborhood N_k $(k=1,2,\ldots,n)$ lies in the preceding neighborhood N_{k-1}.

At this point, Theorem 6.7.3 tells us that $f(z)\equiv 0$ in N_0. But the point z_1 lies in the domain N_0. Hence, by using Theorem 6.7.3 again, $f(z)\equiv 0$ in N_1. Lastly, an application of the same theorem reveals that $f(z)\equiv 0$ in N_n. Since N_n is centered at the point P and since P was arbitrarily selected in D, we may conclude that $f(z)\equiv 0$ in D. This completes the proof of the lemma.

Theorem 6.8.1. *Suppose that two functions* f *and* g *are analytic in the same domain* D *and that* $f(z)=g(z)$ *at each point* z *of some domain or line segment contained in* D*. Then* $h(z)=g(z)$ *throughout* D*.*

Proof. By assumption, we see the difference

$$h(z)=f(z)-g(z)$$

is also analytic in D, and $h(z)=0$ throughout the subdomain or along the line segment.

According to Lemma 6.8.1, then $h(z)=0$ throughout D; that is, $h(z)=g(z)$ at each point z in D. This completes the proof.

This important result tells us that: *a function that is analytic in a domain* D *is uniquely determined over* D *by its values in a domain, or along a line segment, contained in* D.

Also, this theorem is useful in studying the question of extending the domain of definition of an analytic function. More precisely, given two domains D_1 and D_2, consider the intersection $D_1 \cap D_2$. If D_1 and D_2 have points in common (see Fig. 6-10) and a function f_1 is analytic in D_1, there may exist a function f_2, which is analytic in D_2, such that $f_2(z) = f_1(z)$ for each z in the intersection $D_1 \cap D_2$. If so, we call f_2 an analytic continuation of f_1 into the second domain D_2.

Whenever that analytic continuation exists, it is unique, according to Theorem 6.8.1 just proved. That is, not more than one function can be analytic in D_2 and assume the value $f_1(z)$ at each point z of the domain $D_1 \cap D_2$. However, if there is an analytic continuation f_3 of f_2 from D_2 into a domain D_3 which intersects D_1, as indicated in Fig. 6-10, it is not necessarily true that

$$f_3(z) = f_1(z)$$

for each z in $D_1 \cap D_3$.

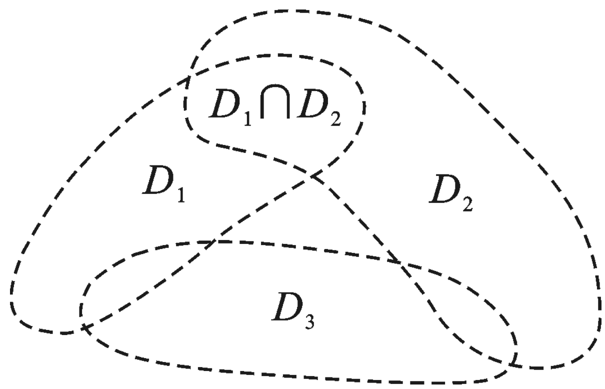

Fig. 6-10

If f_2 is the analytic continuation of f_1 from a domain D_1 into a domain D_2, then the F defined by the equations

$$F(z) = \begin{cases} f_1(z), & z \in D_1, \\ f_2(z), & z \in D_2 \end{cases}$$

is analytic in the union $D_1 \cup D_2$, f_1 and f_2 are called *elements* of F.