§5.6. Absolute and Uniform Convergence of Power Series

This section and the three following it are devoted mainly to various properties of power series. A reader who wishes to simply the theorems and any corollaries there can easily skip their proofs in order to reach Sec. 5.10 more quickly.

We recall from Sec. 5.1 that a series of complex numbers converges absolutely if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

Theorem 5.6.1. *If a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (5.6.1)$$

converges for some point $z = z_1 (z_1 \neq z_0)$, *then it is absolutely convergent at each point* z *in the open disk* $|z - z_0| < R_1$, *where* $R_1 = |z_1 - z_0|$ (Fig. 5-6).

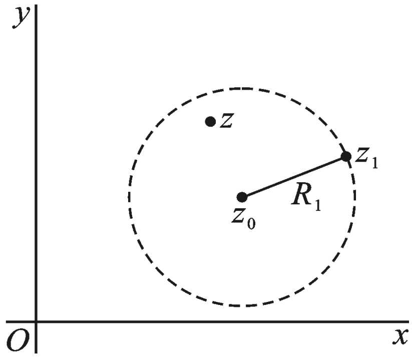

Fig. 5-6

Proof. We assume that $z_1 \neq z_0$ and the series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges. The sequence $\{a_n (z_1 - z_0)^n\}$ is thus bounded; that is,

$$|a_n (z_1 - z_0)^n| \leq M \ (n = 0,1,2,\ldots)$$

for some positive constant M. Let $|z - z_0| < |z_1 - z_0|$ and let $\rho = \left|\frac{z - z_0}{z_1 - z_0}\right|$, we can see that

$$|a_n (z - z_0)^n| = |a_n (z_1 - z_0)^n| \left|\frac{z - z_0}{z_1 - z_0}\right|^n \leq M\rho^n (n = 0,1,2,\ldots). \qquad (5.6.2)$$

Since $0 < \rho < 1$, the series $\sum_{n=0}^{\infty} M\rho^n$ converges. Hence, by comparison test for series of real numbers, the series $\sum_{n=0}^{\infty} |a_n (z - z_0)^n|$ converges. This completes the proof.

Theorem 5.6.1 tells us that the power series (5.6.1) converges inside some circle centered at z_0, provided it converges at some point other than z_0. The greatest circle centered at z_0 such that series (5.6.1) converges at each point inside is called the *disk of convergence* of series (5.6.1) and the radius of the circle is called the *radius of convergence* of the power series (5.6.1). The series cannot converge at any point z_2 outside that circle, according to the theorem; for if it did, it would converge everywhere inside the circle centered at z_0 and passing through z_2. The first circle could not, then, be the disk of convergence. It is easy to check that the radius of convergence R of the series (5.6.1) can be given by

$$R = \sup\{|z_1 - z_0| : \sum_{n=0}^{\infty} a_n (z_1 - z_0)^n \text{ converges}\}.$$

It is obvious that radii of convergence of the following series

$$\sum_{n=0}^{\infty} n! z^n, \sum_{n=0}^{\infty} (z-2)^n, \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

are $0, 1, \infty$, respectively.

Our next theorem involves terminology that we must first define. Suppose that the power series (5.6.1) has disk of convergence $|z - z_0| < R$, and let $S(z)$ and $S_n(z)$ represent the sum and partial sums, respectively, of that series, that is,

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad S_n(z) = \sum_{k=0}^{n-1} a_k (z - z_0)^k \quad (|z - z_0| < R)$$

Then write the remainder function

$$\rho_n(z) = S(z) - S_n(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k \quad (|z - z_0| < R). \tag{5.6.3}$$

Since the power series converges for any fixed value of z when $|z - z_0| < R$, we know that the remainder $\rho_n(z)$ goes zero for any such z as n tends to infinity. According to definition (5.1.2), this means that, for any such z, corresponding to each positive number ε, there is a positive integer N such that

$$|\rho_n(z)| < \varepsilon \text{ whenever } n > N. \tag{5.6.4}$$

Let D be a subset contained in the disk of convergence of (5.6.1), then the series (5.6.1) is said to be *uniformly convergent to* $S(z)$ on D if $\forall \varepsilon > 0, \exists N$ such that whenever $n > N$,

$$|\rho_n(z)| = |S(z) - S_n(z)| < \varepsilon, \forall z \in D.$$

Theorem 5.6.2. *If z_1 is a point inside the disk of convergence $|z - z_0| < R$ of a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{5.6.5}$$

then the series is uniformly convergent on the closed disk $|z - z_0| \le R_1$, where $R_1 = |z_1 - z_0|$ (Fig. 5-7).

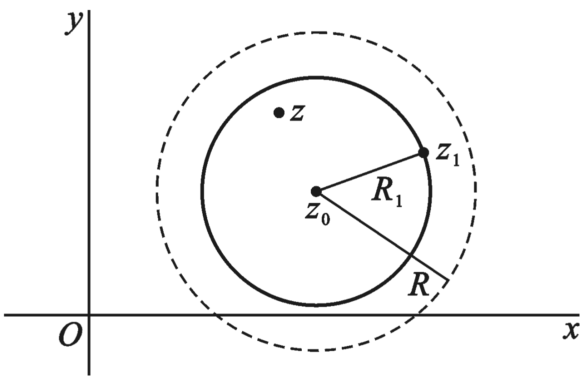

Fig. 5-7

Proof. Given that z_1 is a point in the disk of convergence of the series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n , \tag{5.6.6}$$

according to Theorem 5.6.1, the series

$$\sum_{n=0}^{\infty} \left| a_n (z_1 - z_0)^n \right| \tag{5.6.7}$$

converges. Thus, its remainder $\sum_{k=n}^{\infty} |a_k| \, | R_1 |^k \to 0 (n \to \infty)$. Hence, $\forall \varepsilon > 0, \exists N$ such that

$$\sum_{k=n}^{\infty} |a_k| \, | R_1 |^k < \varepsilon (\forall n > N) . \tag{5.5.8}$$

Thus, when $n > N$, for all $|z - z_0| \le |z_1 - z_0| = R_1$, we have

$$| \rho_n (z) | = \left| \sum_{k=n}^{\infty} a_k (z - z_0)^k \right| \le \sum_{k=n}^{\infty} |a_k| |z - z_0|^k \le \sum_{k=n}^{\infty} |a_k| |R_1|^k < \varepsilon .$$

Hence the series is uniformly convergent on the disk $|z - z_0| \le R_1$. The proof is completed.

From Theorem 5.6.2, we know that if the radius of convergence $R > 0$ of (5.6.5), then for every $0 < R_0 < R$, the power series (5.6.5) is uniformly convergent on the disk $|z - z_0| \le R_0$.

Our last result gives a formula for finding the radius of convergence of a power series.

Theorem 5.6.3. *Let* $L = \lim_{n \to \infty} \frac{| a_{n+1} |}{| a_n |} \in [0, \infty]$, *then the radius of convergence* R *of the power series* (5.6.1) *is* $R = \frac{1}{L}$, *where* $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$.

Proof. Let $z \ne z_0$, then

$$\lim_{n \to \infty} \frac{| a_{n+1} (z - z_0)^{n+1} |}{| a_n (z - z_0)^n |} = L | z - z_0 | .$$

From calculus, we know that $\sum_{n=0}^{\infty} | a_n (z - z_0)^n |$ converges when $L | z - z_0 | < 1$. It follows from Theorem 5.1.2 that the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges when $L | z - z_0 | < 1$. Also, when $L | z - z_0 | > 1$, $\lim_{n \to \infty} | a_n (z - z_0)^n | \ne 0$ and so the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ does not converge.

As a conclusion, when $0 < L < \infty$, $R = \frac{1}{L}$; when $L = 0$, $L | z - z_0 | < 1$ for all $z \in \mathbf{C}$ and so $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for all $z \in \mathbf{C}$, and therefore $R = \infty = \frac{1}{L}$; when $L = \infty$, $L | z - z_0 | > 1$ for all $z \in \mathbf{C} \setminus \{z_0\}$ and so

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

does not converge for all $z \in \mathbf{C} \setminus \{z_0\}$, and therefore $R = 0 = \frac{1}{L}$. This completes the proof.

For example, the radii of convergence of the power series

$$\sum_{n=0}^{\infty} n! (z-1)^n, \sum_{n=0}^{\infty} (n^2 + 3n + 1) z^n, \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}$$

are $0, 1, \infty$, respectively.