§2.4. Limits

In this section, we will discuss the limits of sequences and functions.

1. Definition of a convergent sequence

Let $\{z_n\}$ be a sequence of complex numbers. If there exists a complex number z such that $\lim_{n\to\infty}|z_n - z| = 0$, then we say that $\{z_n\}$ is *convergent* and call the number z to be the *limit* of the sequence $\{z_n\}$, written $\lim_{n\to\infty} z_n = z$, or $z_n \to z(n\to\infty)$.

Proposition 2.4.1. $\lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty}\mathrm{Re}\, z_n = \mathrm{Re}\, z, \lim_{n\to\infty}\mathrm{Im}\, z_n = \mathrm{Im}\, z$.

Proposition 2.4.2. *Let* $\{z_n\},\{w_n\}$ *be sequences of complex numbers.*

(1) *If* $\{z_n\}$ *is convergent, then it is bounded, i.e., there is constant* $M>0$ *such that*

$$|z_n| \le M \quad \textit{for all} \quad n \in N;$$

(2) *If* $\lim_{n\to\infty} z_n = z$ *and* $\lim_{n\to\infty} w_n = w$, *then*

$$\lim_{n\to\infty} cz_n = cz(\forall c\in\mathbf{C}),\ \lim_{n\to\infty}(z_n \pm w_n) = (z\pm w), \textit{ and } \lim_{n\to\infty} z_n w_n = zw;$$

(3) *If* $\lim_{n\to\infty} z_n = z$ *and* $\lim_{n\to\infty} w_n = w \ne 0$, *then*

$$\lim_{n\to\infty} z_n / w_n = z/w.$$

2. Definition of limit of a function

Definition 2.4.2. For a function $f: D\to\mathbf{C}$ and a point z_0, if for each positive number ε, there is a positive number δ such that

$$z\in D, 0<|z-z_0|<\delta \Rightarrow |f(z)-w_0|<\varepsilon, \tag{2.4.1}$$

then we say that w_0 is the *limit* of $f(z)$ as z approaches z_0 and write

$$\lim_{z\to z_0} f(z) = w_0, \text{ or } f(z)\to w_0(z\to z_0). \tag{2.4.2}$$

Geometrically, this definition says that, for each ε-neighborhood

$$N(w_0,\varepsilon) = \{w : |w-w_0|<\varepsilon\}$$

of w_0, there is a deleted δ-neighborhood

$$N^\circ(z_0,\delta) = \{z : 0<|z-z_0|<\delta\}$$

of z_0 such that $f(D\cap N^\circ(z_0,\delta)) \subset N(w_0,\varepsilon)$, see, Fig. 2-8.

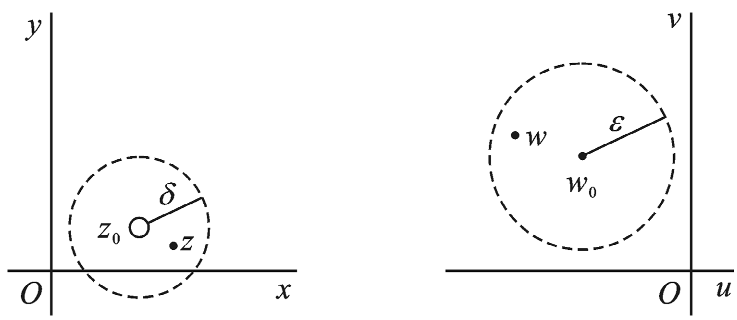

Fig. 2-8

Theorem 2.4.1. *If a limit of a function* f *defined on* D *exists at a point* z_0, *then it is unique.*

Example 1. Let $f(z) = iz/2, D = \{z : |z|<1\}$, then

$$\lim_{z \to 1} f(z) = \frac{i}{2}. \tag{2.4.3}$$

$$\left| f(z) - \frac{i}{2} \right| < \varepsilon \quad \text{whenever} \quad z \in D \text{ and } 0 < |z - 1| < 2\varepsilon.$$

Theorem 2.4.2. *Let* f *be defined on* D*, then* $\lim_{z \to z_0} f(z) = w_0$ *if and only if* $\lim_{n \to \infty} f(z_n) = w_0$ *whenever* $\{z_n\} \subset D \setminus \{z_0\}$ *with* $z_n \to z_0 (n \to \infty)$.

Example 2. If $f(z) = \frac{z}{\bar{z}}$, then the limit $\lim_{z \to 0} f(z)$ does not exist.

Indeed, when $z_n' = (\frac{1}{n}, 0), z_n'' = (0, \frac{1}{n})$, we have

$$z_n' = \tfrac{1}{n} + i0 \to 0, z_n'' = 0 + i\tfrac{1}{n} \to 0 \quad \text{as} \quad n \to \infty.$$

But $\lim_{n \to \infty} f(z_n') = 1$ and $\lim_{n \to \infty} f(z_n'') = -1$. Thus, by Theorem 2.4.2, we know that the limit $\lim_{z \to 0} f(z)$ does not exist. See Fig. 2-10.

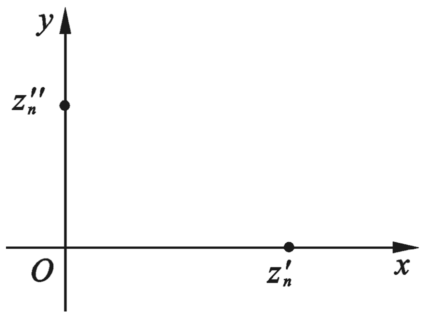

Fig. 2-10

While definition (2.4.1) provides a means of testing whether a given point w_0 is a limit, it does not directly provide a method for determining that limit. Theorems on limits, presented in the next section, will enable us to actually find many limits.