§1.5. Conjugates

1.Conjugate of a complex number

The *conjugate* of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is,

$$\bar{z} = x - iy. \tag{1.5.1}$$

The number $\bar{z}$ is represented by the point $(x,-y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 1-5).

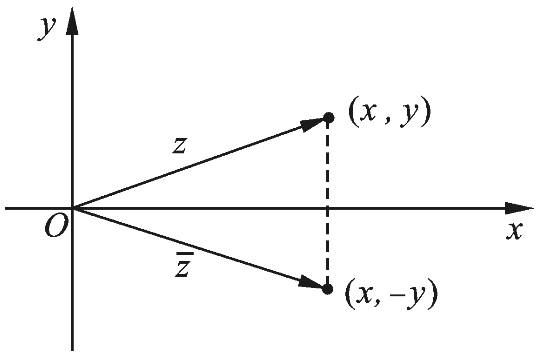

Fig. 1-5

2. Useful identities

$$\bar{\bar{z}} = z \quad , \quad |\bar{z}| = |z|$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2. \tag{1.5.2}$$

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2, \tag{1.5.3}$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \tag{1.5.4}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0). \tag{1.5.5}$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}. \tag{1.5.6}$$

$$z\bar{z} = |z|^2. \tag{1.5.7}$$

Example 1. As an illustration, we compute

$$\frac{-1+3i}{2-i} = \frac{(-1+3i)(2+i)}{(2-i)(2+i)} = \frac{-5+5i}{|2-i|^2} = \frac{-5+5i}{5} = -1 + i.$$

See also the example near the end of Sec.1.3.

Identity (1.5.7) is especially useful in obtaining properties of moduli from properties of conjugates noted above. We mention that

$$|z_1 z_2| = |z_1||z_2| \tag{1.5.8}$$

and

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} (z_2 \neq 0). \tag{1.5.9}$$

Property (1.5.8) can be established by writing

$$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1 z_2)(\bar{z}_1 \bar{z}_2) = (z_1 \bar{z}_1)(z_2 \bar{z}_2) = |z_1|^2 |z_2|^2 = (|z_1||z_2|)^2$$

and recalling that a modulus is never negative. Property (1.5.9) can be verified in a similar way.

Example 2. Property (1.5.8) tells us that $|z|^2=|z|^2$ and $|z|^3=|z|^3$. Hence if z is a point inside the circle centered at the origin with radius 2, so that $|z|<2$, it follows from the generalized form (1.4.9) of the triangle inequality in Sec. 1.4 that

$$|z^3+3z^2-2z+1|\le|z|^3+3|z|^2+2|z|+1<25.$$