## §6.5. Residues at Poles

When a function $f$ has an isolated singularity at a point $z_0$, the basic method for identifying $z_0$ as a pole and finding the residue there is to write the appropriate Laurent series and to find the coefficient of $1/(z-z_0)$. The following theorem provides an alternative characterization of poles and another way of finding the corresponding residues.

**Theorem 6.5.1.** *An isolated singular point $z_0$ of a function $f$ is a pole of order $m$ if and only if $f(z)$ can be written in the form*

$$f(z)=\frac{\phi(z)}{(z-z_0)^m} \quad (0<|z-z_0|<R_2), \tag{6.5.1}$$

*where $\phi(z)$ is analytic and nonzero at $z_0$. Moreover,*

$$\operatorname*{Res}_{z=z_0} f(z)=\frac{\phi^{(m-1)}(z_0)}{(m-1)!} \tag{6.5.2}$$

**Proof.** To prove the theorem, we first assume that $f(z)$ has the form (6.5.1). Since $\phi(z)$ is analytic at $z_0$, it has a Taylor series representation

$$\phi(z)=\phi(z_0)+\frac{\phi'(z_0)}{1!}(z-z_0)+\frac{\phi''(z_0)}{2!}(z-z_0)^2+\cdots+\frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \sum_{n=m}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

in some neighborhood $|z-z_0|<\varepsilon$ of $z_0$ where $\varepsilon<R_2$; and from expression (6.5.1) it follows that

$$f(z)=\frac{\phi(z_0)}{(z-z_0)^m}+\frac{\phi'(z_0)/1!}{(z-z_0)^{m-1}}+\frac{\phi''(z_0)/2!}{(z-z_0)^{m-2}}+\cdots+\frac{\phi^{(m-1)}(z_0)/(m-1)!}{z-z_0} + \sum_{n=m}^{\infty}\frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \tag{6.5.3}$$

when $0<|z-z_0|<\varepsilon$. This Laurent series representation, together with the fact that $\phi(z_0)\neq 0$, reveals that $z_0$ is, indeed, a pole of order $m$ of $f(z)$. The coefficient of $1/(z-z_0)$ tells us, of course, that the residue of $f(z)$ at $z_0$ is as in the statement of the theorem.

Suppose, on the other hand, that we know only that $z_0$ is a pole of order $m$ of $f$, then $f(z)$ has a Laurent series representation

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2} +\cdots+\frac{b_{m-1}}{(z-z_0)^{m-1}}+\frac{b_m}{(z-z_0)^m} \quad (b_m\neq 0),$$

which is valid in a punctured disk $0<|z-z_0|<R_2$. The function $\phi(z)$ defined by means of the equations

$$\phi(z)=\begin{cases}(z-z_0)^m f(z), & \text{when } z\neq z_0,\\ b_m, & \text{when } z=z_0,\end{cases}$$

evidently has the power series representation

$$\phi(z)=b_m+b_{m-1}(z-z_0)+\cdots+b_2(z-z_0)^{m-2}+b_1(z-z_0)^{m-1} + \sum_{n=0}^{\infty}a_n(z-z_0)^{m+n}$$

throughout the entire disk $|z-z_0|<R_2$. Consequently, $\phi(z)$ is analytic in that disk (Sec. 5.8) and, in particular, at $z_0$. Inasmuch as $\phi(z_0)=b_m\neq 0$, expression (6.5.1) is established; and the proof of the theorem is complete.