Chapter V

Series

This chapter is devoted mainly to series representations of analytic functions. We present theorems that guarantee the existence of such representations, and we develop some facility in manipulating series.

§5.1. Convergence of Series

Definition 5.1.1. Let $\{z_n\} \subset \mathbf{C}$ be a sequence, then the following formal notation

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots \tag{5.1.1}$$

is called a *series* of complex numbers, its *n*th partial sum is defined as

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \cdots + z_n \ (n = 1, 2 \ldots). \tag{5.1.2}$$

Definition 5.1.2. The series (5.1.1) is said to be convergent to the sum S if the sequence $\{S_n\}$ of partial sums converges to S; we then write

$$\sum_{n=1}^{\infty} z_n = S.$$

Note that, since a sequence has at most one limit, a series has at most one sum. When a series does not converge, we say that it diverges.

Theorem 5.1.1. *Suppose that*

$$z_n = x_n + iy_n \ (n = 1, 2, \ldots) \quad \textit{and} \quad S = X + iY$$

where x_n, y_n, X, Y *are all real numbers. Then*

$$\sum_{n=1}^{\infty} z_n = S \tag{5.1.3}$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad and \quad \sum_{n=1}^{\infty} y_n = Y . \tag{5.1.4}$$

Proof. To prove the theorem, we first write the partial sums (5.1.2) as

$$S_n = X_n + iY_n , \tag{5.1.5}$$

where

$$X_n = \sum_{k=1}^{n} x_k \quad \text{and} \quad Y_n = \sum_{k=1}^{n} y_k .$$

Then from Proposition 4.2.1, we see that

$$\lim_{n\to\infty} S_n = S \Leftrightarrow \lim_{n\to\infty} X_n = X \quad \text{and} \quad \lim_{n\to\infty} Y_n = Y .$$

Since X_n and Y_n are the *n*th partial sums of the series $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} y_n$, the theorem is proved.

This theorem tells us that one can write

$$\sum_{n=1}^{\infty} (x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n$$

whenever it is known that the two series on the right converge or that the one on the left does.

By recalling from calculus that the *n*th term of a convergent series of real numbers approaches zero as n tends to infinity, we can see immediately from Theorem 5.1.1 that the same is true for a convergent series of complex numbers. That is, a *necessary condition* for the convergence of series (5.1.1) is that

$$\lim_{n\to\infty} z_n = 0 . \tag{5.1.6}$$

Definition 5.1.3. Series (5.1.1) is called *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$$

of real numbers converges.

Since

$$|x_n| \le \sqrt{x_n^2 + y_n^2} = |z_n| \quad \text{and} \quad |y_n| \le \sqrt{x_n^2 + y_n^2} = |z_n| \quad \text{for all} \quad n \in \mathbf{N} ,$$

we know from the *comparison test* in calculus that if the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then two series $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ must converge. Moreover, since the absolute convergence of a series of real numbers implies the convergence of the series itself, it follows that there are real numbers X and Y

to which series (5.1.4) converge. According to Theorem 5.1.1, then, series (5.1.1) converges. Consequently, we get the following result.

Theorem 5.1.2. *Every absolutely convergent series of complex numbers is convergent.*

In establishing the fact that the sum of a series is a given number S, it is often convenient to define the *remainder* ρ_n after n terms:

$$\rho_n = S - S_n . \qquad (5.1.7)$$

Since

$$S = S_n + \rho_n \quad \text{and} \quad |S_n - S| = |\rho_n - 0|,$$

we see that the following conclusion holds.

Theorem 5.1.3. *A series converges to a number* S *if and only if the sequence of remainders tends to zero*.

We shall make considerable use of this theorem in our treatment of power series. They are series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots,$$

where z_0 and the coefficients a_n are complex constants and z may be any point in a stated region containing z_0. In such series, involving a variable z, we shall denote sums, partial sums and remainders by $S(z)$, $S_n(z)$ and $\rho_n(z)$, respectively.

Example. With the aid of remainders, it is easy to verify that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{whenever } |z| < 1. \qquad (5.1.9)$$

We need only recall the identity

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1-z} \ (z \neq 1)$$

to write the partial sums

$$S_n(z) = \sum_{k=0}^{n-1} z^k = 1 + z + z^2 + \cdots + z^{k-1} = \frac{1-z^n}{1-z} \ (z \neq 1) .$$

If $S(z) = \dfrac{1}{1-z}$, then $|\rho_n(z)| = |S(z) - S_n(z)| = \dfrac{|z|^n}{|1-z|} \ (z \neq 1)$.

It is clear from this that the remainder $\rho_N(z)$ tend to zero when $|z| < 1$ but not when $|z| \geq 1$. Summation formula (5.1.9) is established.