§6.6. Examples

The following examples serve to illustrate the use of the theorem in the previous section.

Example 1. The function $f(z) = (z+1)/(z^2+9)$ has an isolated singular point at $z = 3i$ and can be written as

$$f(z) = \frac{\phi(z)}{z-3i} \quad \text{where} \quad \phi(z) = \frac{z+1}{z+3i}.$$

Since $\phi(z)$ is analytic at $z = 3i$ and $\phi(3i) = (3-i)/6 \neq 0$, that point is a simple pole of the function f; and the residue there is $B_1 = (3-i)/6$. The point $z = -3i$ is also a simple pole of f, with residue $B_2 = (3+i)/6$.

Example 2. If $f(z) = (z^3+2z)/(z-i)^3$, then

$$f(z) = \frac{\phi(z)}{(z-i)^3} \quad \text{where} \quad \phi(z) = z^3 + 2z.$$

The function $\phi(z)$ is entire, and $\phi(i) = i \neq 0$. Hence f has a pole of order 3 at $z = i$. The residue there is

$$B = \frac{\phi''(i)}{2!} = 3i.$$

The theorem can, of course, be used when branches of multiple-valued functions are involved.

Example 3. Suppose that

$$f(z) = \frac{(\log z)^3}{z^2+1},$$

where the branch

$$\log z = \ln r + i\theta (r > 0, -\pi < \theta < \pi)$$

of the logarithmic function is to be used. To find the residue of f at $z = i$, we write

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{(\log z)^3}{z+i}.$$

The function $\phi(z)$ is clearly analytic at $z = i$; and, since

$$\phi(i) = \frac{(\log i)^3}{2i} = \frac{(\ln 1 + i\pi/2)^3}{2i} = -\frac{\pi^3}{16} \neq 0,$$

the desired residue is $B = \phi(i) = -\pi^3/16$.

While the theorem in Sec. 6.5 can be extremely useful, the identification of an isolated singular point as a pole of a certain order is sometimes done most efficiently by appealing directly to a Laurent series.

Example 4. If the residue of the function

$$f(z) = \frac{\sinh z}{z^4}$$

is needed at the singular point $z = 0$, it would be incorrect to write

$$f(z) = \frac{\phi(z)}{z^4} \quad \text{where} \quad \phi(z) = \sinh z$$

and to attempt an application of formula (6.5.3) in Sec. 6.5 with $m = 4$. For it is necessary that $\phi(z_0) \neq 0$ if that formula is to be used. In this case, the simplest way to find the residue is to write out a few terms of the Laurent series for $f(z)$, as was done in Example 2 of Sec. 6.4, it was shown that $z = 0$ is a pole of the *third order*, with residue $B = 1/6$.

In some cases, the series approach can be effectively combined with the theorem in Sec. 6.5.

Example 5. Since $z(e^z - 1)$ is entire and its zeros are

$$z = 2n\pi i \ \ (n = 0, \pm 1, \pm 2, \ldots),$$
the point $z = 0$ is clearly an isolated singular point of the function
$$f(z) = \frac{1}{z(e^z - 1)}.$$
From the Maclaurin series
$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad (|z| < \infty),$$
we see that
$$z(e^z - 1) = z\left(\frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right) \quad (|z| < \infty).$$
Thus
$$f(z) = \frac{\phi(z)}{z^2} \ \text{ where } \ \phi(z) = \frac{1}{1 + z/2! + z^2/3! + \cdots}.$$
Since $\phi(z)$ is analytic at $z = 0$ and $\phi(0) = 1 \neq 0$, the point $z = 0$ is a pole of the *second* order; and, according to formula (6.5.3) in Sec. 6.5, the residue is $B = \phi'(0)$. Because
$$\phi'(z) = \frac{-1(1/2! + 2z/3! + \cdots)}{(1 + z/2! + z^2/3! + \cdots)^2}$$
in a neighborhood of the origin, then, $B = -1/2$.

This residue can also be found by dividing the above series representation for $z(e^z - 1)$ into 1, or by multiplying the Laurent series for $1/(e^z - 1)$ in Exercise 3, Sec. 5.11, by $1/z$.