§3.2. The Logarithmic Function

1.Definition of logarithm of a complex number

If w satisfied

$$e^w = z \tag{3.2.1}$$

where z is any nonzero complex number,then w is called a *logarithm* of the number z.

2. Definition of a logarithmic function

The set

$$\mathrm{Log}z = \{\ln|z| + i(\arg z + 2n\pi) : n \in \mathbf{Z}\} = \ln|z| + i\mathrm{Arg}z,\ \forall z \in \mathbf{C} \setminus \{0\} \tag{3.2.2}$$

is called *the logarithm* of z. Usually, we write

$$\mathrm{Log}z = \ln|z| + i(\arg z + 2n\pi)\ (n = 0, \pm 1, \pm 2, \dots),$$

and then get a simple relation

$$e^{\mathrm{Log}z} = \{z\}\ (z \neq 0). \tag{3.2.3}$$

Thus, we get a multi-valued function

$$\mathrm{Log} : \mathbf{C} \setminus \{0\} \to \mathbf{C},$$

called the *logarithmic function*.

3. Examples

Example 1. If $z = -1 - \sqrt{3}i$, then $r = 2$ and $\theta = -2\pi/3$. Hence

$$\mathrm{Log}(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = \ln 2 + 2\left(n - \frac{1}{3}\right)\pi i\ (n = 0, \pm 1, \pm 2, \dots).$$

Equality (3.2.3) is valid for all nonzero complex number, but the equality $\mathrm{Log}e^z = z$ is not true. To find $\mathrm{Log}e^z = z$, we use the definition of e^z (Sec. 3.1) and see that

$$|e^z| = e^x \text{ and } \mathrm{Arg}(e^z) = y + 2n\pi\ (n = 0, \pm 1, \pm 2, \dots)$$

when $z = x + iy$. Hence, we know that

$$\mathrm{Log}(e^z) = \ln|e^z| + i\mathrm{Arg}(e^z) = \ln(e^x) + i(y + 2n\pi) = (x + iy) + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \dots).$$

Therefore,

$$\mathrm{Log}(e^z) = z + 2n\pi i\ (n = 0, \pm 1, \pm 2, \dots). \tag{3.2.4}$$

The *principal value* of $\mathrm{Log}z$ is the value obtained from equation (3.2.2) when $n = 0$ there and is denoted by $\log z$. Thus

$$\log z = \ln r + i \arg z. \tag{3.2.5}$$

Note that $\log z$ is well defined and single-valued when $z \neq 0$ and that

$$\mathrm{Log}z = \log z + 2n\pi i\ (n = 0, \pm 1 \pm 2, \dots). \tag{3.2.6}$$

Clearly, $\log z$ reduces to the usual logarithm in calculus when z is a positive real number $z = r$. To see this, one need only write $z = re^{i0}$, in which case equation (3.2.5) becomes $\log z = \ln r$. That is, $\log r = \ln r$.

Example 2. From expression (3.2.2), we find that

$$\mathrm{Log}1 = \ln 1 + i(0 + 2n\pi) = 2n\pi i\ (n = 0, \pm 1, \pm 2, \dots).$$

As expected, $\log 1 = 0$.

Our final example here reminds us that, although we were unable to find logarithms of negative real numbers in calculus, we can now do so.

Example 3. Observe that

$$\mathrm{Log}(-1) = \ln 1 + i(\pi + 2n\pi) = (2n + 1)\pi i\ (n = 0, \pm 1, \pm 2, \dots)$$

and that $\log(-1) = \pi i$.