§1.9. Examples

In each of the examples here, we start with expression (1.8.3), Sec. 1.8, and proceed in the manner described at the end of that section.

Example 1. In order to determine the *n*th roots of unity, we write

$$1 = 1\exp[i(0+2k\pi)] \; (k = 0,1,2,\ldots,n-1)$$

and find that

$$1^{1/n} = \sqrt[n]{1}\exp\left[i\left(\frac{0}{n}+\frac{2k\pi}{n}\right)\right] = \exp\left(i\frac{2k\pi}{n}\right) \; (k = 0,1,2,\ldots,n-1). \qquad (1.9.1)$$

When $n = 2$, these roots are, of course, ± 1. When $n \ge 3$, the regular polygon at whose vertices the roots lie is inscribed in the unit circle $|z| = 1$, with one vertex corresponding to the principal root $z = 1(k = 0)$. If we write

$$\omega_n = \exp\left(i\frac{2\pi}{n}\right), \qquad (1.9.2)$$

it follows from property (1.7.8), Sec.1.7, of $e^{i\theta}$ that

$$\omega_n^k = \exp\left(i\frac{2k\pi}{n}\right) \; (k = 0,1,2,\ldots,n-1).$$

Hence, the distinct *n*th roots of unity just found are simply $1,\omega_n,\omega_n^2,\ldots,\omega_n^{n-1}$. See Fig. 1-12, where the cases $n = 3,4$, and 6 are illustrated. Note that $\omega_n^n = 1$.

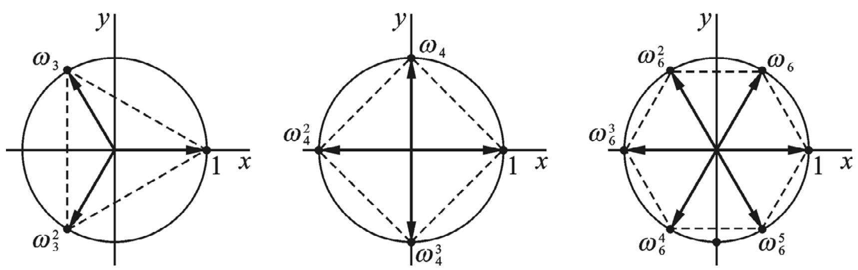

Fig. 1-12

Finally, it is worthwhile observing that if c is any particular *n*th root of z_0, then the set of *n*th roots of z_0 can be put in the form

$$c, c\omega_n, c\omega_n^2, \ldots, c\omega_n^{n-1}.$$

This is because multiplication of any nonzero complex number by ω_n increases the argument of that number by $2\pi/n$, while leaving its modulus unchanged.

Example 2. Let us find all values of $(-8i)^{1/3}$, that is, the three cube roots of $-8i$. To do this, we write

$$-8i = 8\exp\left[i\left(-\frac{\pi}{2}+2k\pi\right)\right] \; (k = 0,\pm1,\pm2,\ldots)$$

and see that the desired roots are

$$c_k = 2\exp\left[i\left(-\frac{\pi}{6}+\frac{2k\pi}{3}\right)\right] \; (k = 0,1,2). \qquad (1.9.3)$$

They lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$. Clearly,

$$c_0 = 2\exp\left[i\left(-\frac{\pi}{6}\right)\right] = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \sqrt{3} - i\,.$$

Without any further calculations, it is then evident that $c_1 = 2i$; and, since c_2 is symmetric to c_0 with respect to the imaginary axis, we know that $c_2 = -\sqrt{3} + i$.

These roots can, of course, be written $c_0, c_0\omega_3, c_0\omega_3^2$, where $\omega_3 = \exp(i2\pi/3)$. (See the remarks at the end of Example 1.)

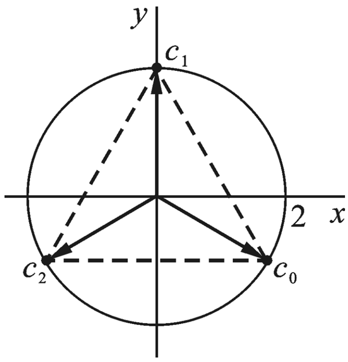

Fig. 1-13

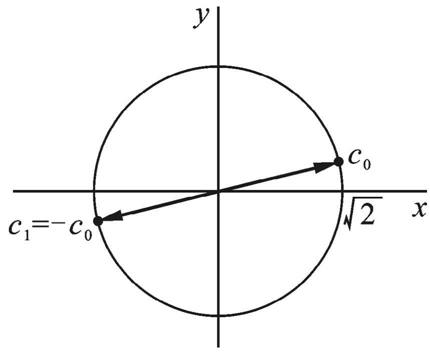

Fig. 1-14

Example 3. The two values $c_k\,(k = 0,1)$ of $(\sqrt{3} + i)^{1/2}$, which are the square roots of $(\sqrt{3} + i)$, are found by writing

$$\sqrt{3} + i = 2\exp\left[i\left(\frac{\pi}{6} + 2k\pi\right)\right] \quad (k = 0, \pm 1, \pm 2, \ldots)$$

and (see Fig. 1-14)

$$c_k = \sqrt{2}\exp\left[i\left(\frac{\pi}{12} + k\pi\right)\right] \quad (k = 0,1)\,. \tag{1.9.4}$$

Euler's formula (See. 1.6) tells us that

$$c_0 = \sqrt{2}\exp\left(i\frac{\pi}{12}\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right),$$

and the trigonometric identities

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos\alpha}{2}, \quad \sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{2} \tag{1.9.5}$$

enable us to write

$$\cos^2\frac{\pi}{12} = \frac{1}{2}\left(1 + \cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{2 + \sqrt{3}}{4}$$

$$\sin^2\frac{\pi}{12} = \frac{1}{2}\left(1 - \cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2 - \sqrt{3}}{4}\,.$$

Consequently,

$$c_0 = \sqrt{2}\left(\sqrt{\frac{2 + \sqrt{3}}{4}} + i\sqrt{\frac{2 - \sqrt{3}}{4}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2 + \sqrt{3}} + i\sqrt{2 - \sqrt{3}}\right).$$

Since $c_1 = -c_0$, the two square roots of $\sqrt{3} + i$ are

$$\pm \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}} \right).$$

ube roots are, in rectangular form, the numbers

$$c_0\omega_3 = \frac{-(\sqrt{3}+1)+(\sqrt{3}-1)i}{\sqrt{2}},$$

$$c_0\omega_3^2 = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}.$$