§1.10. Regions in the Complex Plane

In this section, we are concerned with sets of complex numbers, or points in the z -plane, and their closeness to one another.

1. Definiton of neighborhood of a complex number

An ε *-neighborhood of* z_0 *is defined as*

$$N(z_0,\varepsilon) = \{z : |z - z_0| < \varepsilon\} \tag{1.10.1}$$

where z_0 is a given point.

When the value of ε is understood or is immaterial in the discussion, the set (1.10.1) is often referred to as just a neighborhood.

2. Definiton of deleted neighborhood of a complex number

An *deleted neighborhood of* z_0 *is defined as*

$$N^{\circ}(z_0,\varepsilon) = \{z : 0 < |z - z_0| < \varepsilon\}, \tag{1.10.2}$$

where z_0 is a given point.

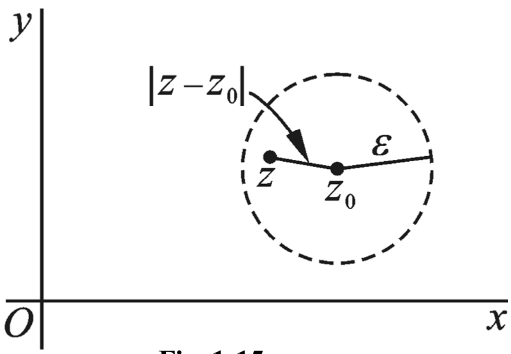

Fig. 1-15

2.Some concepts relating points

A point $z_0 \in \mathbf{C}$ is said to be an *interior point* of a set $S \subset \mathbf{C}$ whenever there is some neighborhood of z_0 that contains only points of S; it is called an *exterior point* of S when there exists a neighborhood of it containing no points of S. The set of all interior (resp. exterior) points of a set S is called the *interior* (resp. *exterior*) of S and denoted by S° (resp. S^e). If z_0 is neither an interior point nor an exterior point of S, it is a *boundary point* of S. The set of all boundary points of a set S is called the *boundary* of S and denoted by S^b or ∂S. A boundary point is, therefore, a point all of whose neighborhoods contain points in S and points not in S. The unit circle $T = \{z : |z| = 1\}$, for instance, is the boundary of each of the sets

$$D = \{z : |z| < 1\} \text{ and } \overline{D} = \{z : |z| \le 1\}. \tag{1.10.3}$$

It is easy to see that for any subset S of the complex plane $\mathbf{C}$, we have

$$\mathbf{C} = S^{\circ} \cup S^b \cup S^e.$$

3. Definiton of an open set

A set is said to be *open* if it contains none of its boundary points, i.e.,

$$S \bigcap S^b = \phi .$$

Proposition 1.10.1. *A set* $S \subset \mathbf{C}$ *is open if and only if each of its points is an interior point,* i.e., $S = S^\circ$.

4.Definiton of a closed set

A set $S \subset \mathbf{C}$ is said to be *closed* if it contains all of its boundary points, i.e., $S \supset S^b$; and the *closure* of a set S , denoted by $\overline{S}$, is the set consisting of all points in S together with the boundary of S , i.e., $\overline{S} = S \bigcup S^b$.

Proposition 1.10.2. *A set* $S \subset \mathbf{C}$ *is closed if and only if* $S = \overline{S}$.

5. Definiton of a connected set

A set $S \subset \mathbf{C}$ is called *connected* if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S . The set $\overline{D} = \{z : |z| \leq 1\}$ is connected. The annulus $\{z : 1 < |z| \leq 2\}$ is also connected (see Fig. 1-16).

Fig. 1-16

6. Definiton of a domain

An open set that is connected is called a *domain*. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a *region*. Thus, a subset S of the plane $\mathbf{C}$ is a region if and only if there are a domain D and a subset E of the boundary D^b such that $S = D \bigcup E$.

7. Definiton of a bounded set

A set S is called *bounded* if all points of S lie inside some circle $|z| = R$, i.e.,

$$S \subset \{z : |z| \leq R\};$$

otherwise, it is called *unbounded.* Both of the sets (1.10.3) are bounded regions, and the right half plane $\{z : \operatorname{Re} z \geq 0\}$ is unbounded.

8. Definiton of a accumulation point

A point z_0 is said to be an *accumulation point* of a set S if each deleted neighborhood of z_0 contains at least one point of S. In other words, a point z_0 is an accumulation point of a set S if

$$S \bigcap N^{\circ}(z_0, \delta) \neq \phi$$

for all positive number δ.

Proposition 1.10.3. *A set $S \subset \mathbf{C}$ is closed if and only if it contains all of its accumulation points.*

Proposition 1.10.4. *A set $S \subset \mathbf{C}$ is closed if and only if its complement $S^c = \mathbf{C} \setminus S$ with respective to $\mathbf{C}$ is open.*