§1.7. Products and Quotients in Exponential Form

1. Products in Exponential Form

.If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$,then the product $z_1 z_2$ has exponential form

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} . \qquad (1.7.1)$$

2. Quotients in Exponential Form

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \frac{e^{i\theta_1} e^{-i\theta_2}}{e^{i\theta_2} e^{-i\theta_2}} = \frac{r_1}{r_2} \cdot \frac{e^{i(\theta_1-\theta_2)}}{e^{i0}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} . \qquad (1.7.2)$$

it is easy to find that the inverse of any nonzero complex number $z = re^{i\theta}$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{r} e^{-i\theta} . \qquad (1.7.3)$$

3. An important identity

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 . \qquad (1.7.4)$$

Statement (1.7.4) is not valid when Arg is replaced everywhere by arg. See the following example.

Example 1. When $z_1 = -1$ and $z_2 = i$, we have

$$\arg(z_1 z_2) = \arg(-i) = -\frac{\pi}{2}, \text{ but } \arg z_1 + \arg z_2 = \pi + \frac{\pi}{2} = \frac{3\pi}{2} .$$

Fig. 1-9

Statement (1.7.4) tells us that

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1 z_2^{-1}) = \operatorname{Arg} z_1 + \operatorname{Arg}(z_2^{-1}) , \quad z_1 z_2 \neq 0 .$$

and we can see from expression (1.7.3) that if $\theta_2 \in \operatorname{Arg} z_2$, then

$$-\theta_2 \in \operatorname{Arg}(1/z_2) ,$$

and so

$$\operatorname{Arg}(z_2^{-1}) = \{-\theta_2 + 2n\pi : n \in \mathbf{Z}\} = -\{\theta_2 + 2m\pi : m \in \mathbf{Z}\} = -\operatorname{Arg} z_2 .$$

This shows that

$$\operatorname{Arg}(z_2^{-1}) = -\operatorname{Arg} z_2 . \qquad (1.7.5)$$

Hence (See, Exercise 11, Sec. 1.2)

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 . \qquad (1.7.6)$$

Example 2. In order to find the principal argument $\arg z$ when $z = \frac{-2}{1+\sqrt{3}i}$, observe that

$$\operatorname{Arg} z = \operatorname{Arg}(-2) - \operatorname{Arg}(1+\sqrt{3}i) .$$

Since

$$\arg(-2) = \pi \quad \text{and} \quad \arg(1+\sqrt{3}i) = \frac{\pi}{3} ,$$

one value of $\operatorname{Arg} z$ is $2\pi/3$; and, because $2\pi/3$ is between $-\pi$ and π , we find that

$\arg z = 2\pi/3$.

Another important result that can be obtained formally by applying rules for real number to $z = re^{i\theta}$ is

$$z^n = r^n e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{1.7.7}$$

It is easily verified for positive values of n by mathematical induction. To be specific, we first note that if becomes $z = re^{i\theta}$ when $n = 1$. Next, we assume that it is valid when $n = m$, where m is any positive integer. In view of expression (1.7.1) for the product of two nonzero complex numbers in exponential form, it is then valid for $n = m + 1$:

$$z^{m+1} = zz^m = re^{i\theta} r^m e^{im\theta} = r^{m+1} e^{i(m+1)\theta}.$$

Expression (1.7.7) is thus verified when n is a positive integer. It also holds when $n = 0$, with the convention that $z^0 = 1$. If $n = -1, -2, \ldots,$ on the other hand, we define z^n in terms of the multiplicative inverse of z by writing.

$$z^n = (z^{-1})^m \quad \text{where} \quad m = -n = 1, 2, \ldots.$$

Then, since expression (1.7.7) is valid for positive integral powers, it follows from the exponential form (1.7.3) of z^{-1} that

$$z^n = \left[\frac{1}{r} e^{i(-\theta)}\right]^m = \left(\frac{1}{r}\right)^m e^{im(-\theta)} = \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} = r^n e^{in\theta} \quad (n = -1, -2, \ldots)$$

Expression (1.7.7) is now established for all integral powers.

Observe that if $r = 1$, expression (1.7.7) becomes

$$(e^{i\theta})^n = e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \ldots) \tag{1.7.8}$$

When written in the form

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n = 0, \pm 1, \pm 2, \ldots), \tag{1.7.9}$$

this is known as *de Moivre's formula*.

Expression (1.7.7) can be useful in finding powers of complex numbers even when they are given in rectangular form and the result is desired in that form.

Example 3. In order to put $(\sqrt{3} + i)^7$ in rectangular form, one need only write

$$(\sqrt{3} + i)^7 = (2e^{i\pi/6})^7 = 2^7 e^{i7\pi/6} = (2^6 e^{i\pi})(2e^{i\pi/6}) = -64(\sqrt{3} + i).$$