§5.2. Taylor Series

We turn now to Taylor's theorem, which is one of the most important results of the chapter.

Theorem 5.2.1(Taylor). *Suppose that* f *is analytic in a disk* $|z-z_0|<R_0$ (Fig. 5-1). *Then* $f(z)$ *has the power series representation*

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n \quad (|z-z_0|<R_0), \tag{5.2.1}$$

where

$$a_n=\frac{f^{(n)}(z_0)}{n!} \quad (n=0,1,2,\ldots). \tag{5.2.2}$$

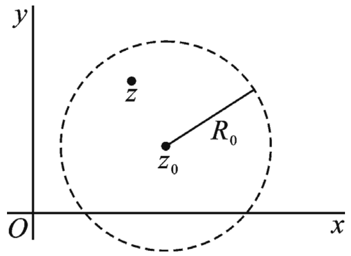

Fig. 5-1

Proof. We first prove the theorem when $z_0=0$, in which case series (5.2.1) becomes

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n \quad (|z|<R_0) \tag{5.2.3}$$

and is called a *Maclaurin series*.

To prove (5.2.3), we let $|z|<R_0$ and write $|z|=r$. Let C_0 denote any positively oriented circle $|z|=r_0$, where $r<r_0<R_0$ (See Fig. 5-2). Since f is analytic inside and on the circle C_0 and the point z is interior to C_0, the Cauchy integral formula yields that

$$f(z)=\frac{1}{2\pi i}\int_{C_0}\frac{f(s)ds}{s-z}. \tag{5.2.4}$$

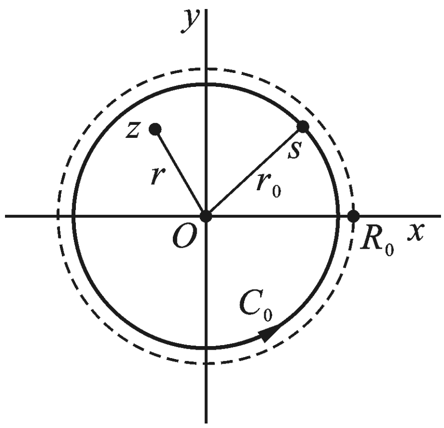

Fig. 5-2

Now the factor $1/(s-z)$ in the integrand here can be put in the form

$$\frac{1}{s-z}=\frac{1}{s}\cdot\frac{1}{1-(z/s)}; \tag{5.2.5}$$

and we know from the example in Sec. 5.2 that

$$\frac{1}{1-z}=\sum_{k=0}^{n-1} z^k+\frac{z^n}{1-z} \tag{5.2.6}$$

when z is any complex number other than unity. Replacing z by z/s in expression (5.2.7), then, we can rewrite equation (5.2.5) as

$$\frac{1}{s-z}=\sum_{k=0}^{n-1}\frac{1}{s^{k+1}}z^k+z^n\frac{1}{(s-z)s^n}. \tag{5.2.8}$$

Multiplying through this equation by $f(s)$ and then integrating each side with respect to s around C_0, we find that

$$\int_{C_0}\frac{f(s)ds}{s-z}=\sum_{k=0}^{n-1}\int_{C_0}\frac{f(s)ds}{s^{k+1}}z^k+z^n\int_{C_0}\frac{f(s)ds}{(s-z)s^n}.$$

In view of expression (5.2.4) and the fact (Sec. 4.13) that

$$\frac{1}{2\pi i}\int_{C_0}\frac{f(s)ds}{s^{k+1}}=\frac{f^{(k)}(0)}{k!}\ (k=0,1,2,\ldots),$$

this reduces, after we multiply through by $1/(2\pi i)$, to

$$f(z)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}z^k+\rho_n(z), \tag{5.2.9}$$

where

$$\rho_n(z)=\frac{z^n}{2\pi i}\int_{C_0}\frac{f(s)ds}{(s-z)s^n}. \tag{5.2.10}$$

Representation (5.2.3) now follows once it is shown that

$$\lim_{n\to\infty}\rho_n(z)=0. \tag{5.2.11}$$

To accomplish this, we recall that $|z|=r$ and that C_0 has radius r_0, where $r_0>r$. Then, if s is a point on C_0, we can see that $|s-z|\geq|s|-|z|=r_0-r$. Consequently, if M denotes the maximum value of $|f(s)|$ on C_0, then

$$|\rho_n(z)|\leq\frac{r^n}{2\pi}\cdot\frac{M}{(r_0-r)r_0^n}2\pi r_0=\frac{Mr_0}{r_0-r}\left(\frac{r}{r_0}\right)^n\to 0(n\to\infty),$$

because of $(r/r_0)<1$. Limit (5.2.11) clearly holds.

To verify the theorem when the disk of radius R_0 is centered at an arbitrary point z_0, we suppose that f is analytic when $|z-z_0|<R_0$ and note that the composite function $f(z+z_0)$ must be analytic when $|(z+z_0)-z_0|<R_0$. This last inequality is, of course, just $|z|<R_0$; and, if we writ $g(z)=f(z+z_0)$, the analyticity of g in the disk $|z|<R_0$ ensures the existence of a Maclaurin series representation:

$$g(z)=\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}z^n\quad(|z|<R_0).$$

That is,

$$f(z+z_0)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}z^n\quad(|z|<R_0).$$

After replacing z by $z - z_0$ in this equation and its condition of validity, we have the desired the expansion (5.2.1). The proof is completed.

Note that the series (5.2.1) can be written

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots (|z - z_0| < R_0), \tag{5.2.12}$$

which is called the *Taylor expansion* of $f(z)$ about the point z_0. And the series on the right hand side of (5.2.12) is called the Taylor series about z_0. It is the familiar Taylor series from calculus, adapted to functions of a complex variable.

From the Taylor's theorem above, we know that any function which is analytic at a point z_0 can be expanded as a Taylor series about z_0. For, if f is analytic at z_0, then it is analytic throughout some neighborhood $|z - z_0| < R_0$ of that point (Sec. 2.13) and then has the Taylor expansion. Also, if f is entire, R_0 can be chosen arbitrarily large; and the condition of validity becomes $|z - z_0| < \infty$. The series then converges to $f(z)$ at each point z in the finite plane.