§4.9. Cauchy Integral Theorem

In Sec. 4.7, we saw that when a continuous function f has a primitive function in a domain D, the integral of f around any given closed path C lying entirely in D has value zero. In this section, we present a theorem, called Cauchy Integral Theorem, giving other conditions on a function f, which ensure that the value of the integral of f around a *simple* closed path is zero. The theorem is central to the theory of functions of a complex variable; and some extensions of it, involving certain special types of domains, will be given in Sec. 4.11.

Lemma 4.9.1. *Let f be piecewise continuous on a path $C: z = z(t)$ $(a \le t \le b)$, described in the positive sense* (counterclockwise), then

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy,$$

where $f(z) = u(x,y) + iv(x,y)$.

Proof. We may assume that f is continuous on C. According to the definition of the integral, we have

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt. \tag{4.9.1}$$

We write $z(t) = x(t) + iy(t)$, then the integrand $f[z(t)]z'(t)$ in expression (4.9.1) is the product of the functions

$$u[x(t), y(t)] + iv[x(t), y(t)] \text{ and } x'(t) + iy'(t)$$

of the real variable t. Thus

$$\begin{aligned}\int_C f(z)dz = &\int_a^b [u(x(t), y(t))x'(t) - v(x(t), y(t))y'(t)]dt \\ &+ i\int_a^b [v(x(t), y(t))x'(t) + u(x(t), y(t))y'(t)]dt.\end{aligned} \tag{4.9.2}$$

In terms of line integrals of real-valued functions of two real variables, we have

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy. \tag{4.9.3}$$

This completes the proof.

Observe that expression (4.9.3) can be obtained formally by replacing $f(z)$ and dz on the left with the binomials $u + iv$ and $dx + idy$, respectively, and expanding their product.

We next recall a result from calculus that enables us to express the line integrals on the right in equation (4.9.3) as double integrals. Suppose that two real-valued functions $P(x,y)$ and $Q(x,y)$ have continuous first-order partial derivatives throughout the closed region $R = \overline{\text{ins}C}$. According to *Green's theorem,* we obtain

$$\int_C Pdx + Qdy = \iint_R (Q_x - P_y)dxdy.$$

With Lemma 4.9.1, we can prove the following result which will be generalized as the so-called Cauchy Integral Theorem.

Proposition 4.9.1. *If C is a simple closed path and a function f is analytic on $R = \overline{\text{ins}(C)}$ and f' is continuous there, then*

$$\int_C f(z)dz = 0. \tag{4.9.4}$$

Proof. Write $f(z) = u(x,y) + iv(x,y) \equiv u + iv$, then from Lemma 4.9.1, we get (4.9.3) hold. Since f is is analytic on $R = \overline{\text{ins}(C)}$, the functions u and v are also continuous on R. Likewise, if the derivative f' of f is continuous on R, so are the first-order partial derivatives of u and v. Green's theorem then enables us to rewrite equation (4.9.3) as

$$\int_C f(z)dz = \iint_R (-v_x - u_y)dxdy + i\iint_R (u_x - v_y)dxdy. \qquad (4.9.5)$$

But, in view of the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$, the integrands of these two double integrals are zero throughout R. So, (4.9.4) is valid. This completes the proof.

Note that, once it has been established that the value of this integral is zero, the orientation of C is immaterial. That is, statement (4.9.4) is also true if C is taken in the clockwise direction, since

$$\int_C f(z)dz = -\int_{-C} f(z)dz = 0.$$

Example 1. If C is any simple closed path, in either direction, then

$$\int_C \exp(z^3)dz = 0.$$

This is because the function $f(z) = \exp(z^3)$ is analytic everywhere and its derivative $f'(z) = 3z^2 \exp(z^3)$ is continuous everywhere.

This result was obtained by Cauchy in the early part of the nineteenth century. Goursat was the first to prove that *the condition of continuity on* f' *can be omitted.* We now state the revised form, known as the *Cauchy Integral Theorem.*

Theorem 4.9.1(Cauchy Integral Theorem, CIT). *If* C *is a simple closed path and a function* f *is analytic on* $R = \overline{\text{ins}(C)}$*, then* $\int_C f(z)dz = 0$.

The proof is presented in the next section, where, to be specific, we assume that C is positively oriented. The reader who wishes to accept this theorem without proof may pass directly to Sec. 4.11.

Example 2. If C is the unit circle $|z| = 1$, then

$$\int_C (\sin z + e^z \cos z)dz = 0.$$

This is because that the integrand $f(z) = \sin z + e^z \cos z$ is analytic on the closed unit disk.