§3.6. Trigonometric Functions

1.Definition of sine and cosine functions of a complex variable

The *sine* and *cosine* functions of a complex variable z as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \forall z \in \mathbf{C}. \qquad (3.6.1)$$

These functions are entire since they are linear combinations of the entire functions e^{iz} and e^{-iz}. Knowing the derivatives of those exponential functions, we find from equations (3.6.1) that

$$\frac{d}{dz}\sin z = \cos z, \quad \frac{d}{dz}\cos z = -\sin z, \quad \forall z \in \mathbf{C}. \qquad (3.6.2)$$

It is easy to see that

$$\sin(-z) = -\sin z \text{ and } \cos(-z) = \cos z, \quad \forall z \in \mathbf{C}; \qquad (3.6.3)$$

and a variety of other identities from trigonometry are valid with complex variable.

Example. In order to show that

$$2\sin z_1 \cos z_2 = \sin(z_1 + z_2) + \sin(z_1 - z_2), \quad \forall z_1, z_2 \in \mathbf{C}, \qquad (3.6.4)$$

using definitions (3.6.1) and properties of the exponential function, we first write

$$2\sin z_1 \cos z_2 = 2\left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right).$$

Multiplication then reduces the right-hand side here to

$$\frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} + \frac{e^{i(z_1-z_2)} - e^{-i(z_1-z_2)}}{2i},$$

that is, $\sin(z_1 + z_2) + \sin(z_1 - z_2)$; and identity (3.6.4) is established.

2.Useful eqautions

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2, \qquad (3.6.5)$$

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \qquad (3.6.6)$$

$$\sin^2 z + \cos^2 z = 1, \qquad (3.6.7)$$

$$\sin 2z = 2\sin z \cos z, \quad \cos 2z = \cos^2 z - \sin^2 z, \qquad (3.6.8)$$

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \sin\left(z - \frac{\pi}{2}\right) = -\cos z. \qquad (3.6.9)$$

$$\sin(iy) = i\sinh y \text{ and } \cos(iy) = \cosh y, \qquad (3.6.10)$$

where $\sinh y = \frac{e^y - e^{-y}}{2}, \cosh y = \frac{e^y + e^{-y}}{2}$

$$\sin z = \sin x \cosh y + i\cos x \sinh y, \qquad (3.6.11)$$

$$\cos z = \cos x \cosh y - i\sin x \sinh y, \qquad (3.6.12)$$

where $z = x + iy$.

$$\sin(z + 2\pi) = \sin z, \quad \sin(z + \pi) = -\sin z, \qquad (3.6.13)$$

$$\cos(z + 2\pi) = \cos z, \quad \cos(z + \pi) = -\cos z. \qquad (3.6.14$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \qquad (3.6.15)$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \qquad (3.6.16)$$

3.Definition of zero of an analytic function

A *zero* of an analytic function f is a number z_0 such that $f(z_0) = 0$.

$$\sin z = 0 \quad \textit{if and only if} \quad z = n\pi (n = 0, \pm 1, \pm 2, \ldots),$$

$$\cos z = 0 \quad \textit{if and only if} \quad z = \frac{\pi}{2} + n\pi (n = 0, \pm 1, \pm 2, \ldots).$$

4.Definitions of other trigonometric functions

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}. \tag{3.6.17}$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}. \tag{3.6.18}$$

5. Derivatives of other trigonometric functions

$$\frac{d}{dz}\tan z = \sec^2 z, \quad \frac{d}{dz}\cot z = -\csc^2 z, \tag{3.6.19}$$

$$\frac{d}{dz}\sec z = \sec z \tan z, \quad \frac{d}{dz}\csc z = -\csc z \cot z. \tag{3.6.20}$$