§7.8. Argument Principle

A function f is said to be *meromorphic* in a domain D if it is analytic throughout D except for possible poles. Suppose now that f is meromorphic in the domain interior to a positively oriented simple closed contour C and that it is analytic and nonzero on C. The image Γ of C under the transformation $w = f(z)$ is a closed contour, not necessarily simple, in the w plane (Fig. 7-10). As a point z traverses C in the positive direction, its images w traverses Γ in a particular direction that determines the orientation of Γ. Note that, since f has no zeros on C, the contour Γ does not pass through the origin in the w plane.

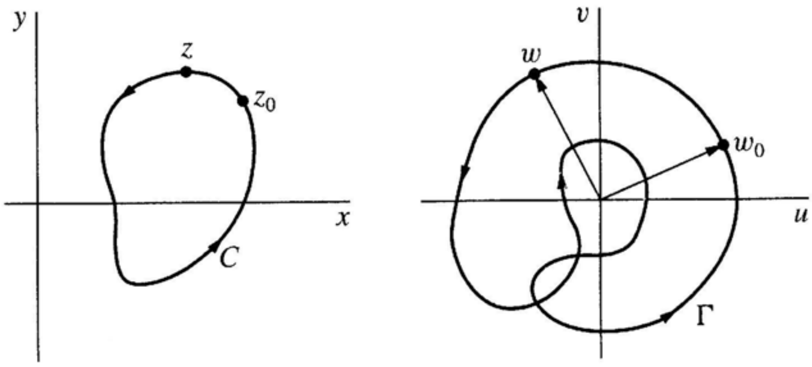

Fig. 7-10

Let w and w_0 be points on Γ, where w_0 is fixed and ϕ_0 is a value of $\text{Arg}w_0$. Then let $\arg w$ vary continuously, starting with the value ϕ_0, as the point w begins at the point w_0 and traverses Γ once in the direction of orientation assigned to it by the mapping $w = f(z)$. When w returns to the point w_0, where it started, $\arg w$ assumes a particular value of $\arg w_0$, which we denote by ϕ_1. Thus the change in $\arg w$ as w describes Γ once in its direction of orientation is $\phi_1 - \phi_0$. This change is, of course, independent of the point w_0 chosen to determine it. Since $w = f(z)$, the number $\phi_1 - \phi_0$ is, in fact, the change in argument of $f(z)$ as z describes C once in the positive direction, starting with a point z_0; and we write

$$\Delta_C \arg f(z) = \phi_1 - \phi_0 .$$

The value of $\Delta_C \arg f(z)$ is evidently an integral multiple of 2π, and the integer

$$\frac{1}{2\pi}\Delta_C \arg f(z)$$

represents the number of times that the point w winds around the origin in the plane. For that reason, this integer is sometimes called the *winding number* of Γ with respect to the origin $w = 0$. It is positive if Γ winds around the origin in the counterclockwise direction and negative if it winds clockwise around that point. The winding number is always zero when Γ does not enclose the origin. The verification of this fact for a special case is left to the exercises.

The winding number can be determined from the number of zeros and poles of f interior to C. The number of poles is necessarily finite since the accumulation points of the poles must not isolated singular points. Likewise, it is easily shown (Exercise 3, Sec. 7.8) that the zeros of f are finite in number. Suppose now that f has Z zeros and P poles in the domain interior to

C, where we agree that f has m_0 zeros at a point z_0 if it has a zero of order m_0 there; and if f has a pole of order m_p at z_0, that pole is to be counted m_p times. The following theorem, which is known as the *argument principle*, states that the winding number is simply the difference $Z-P$.

Theorem 7.8.1. *Suppose that*

(i) *a function* $f(z)$ *is meromorphic in the domain interior to a positively oriented simple closed contour* C*;*

(ii) $f(z)$ *is analytic and nonzero on* C*;*

(iii) *counting multiplicities,* Z *is the number of zeros and* P *is the number of poles of* $f(z)$ *inside* C*.*

Then

$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z - P. \tag{7.8.1}$$

Proof. To prove this, we evaluate the integral of $f'(z)/f(z)$ around C in two different ways. First, we let $z = z(t)$ $(a \le t \le b)$ be a parametric representation for C, so that

$$\int_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{f'[z(t)]z'(t)}{f[z(t)]}dt. \tag{7.8.2}$$

Since, under the transformation $w = f(z)$, the image Γ of C never passes through the origin in the w plane, the image of any point $z = z(t)$ on C can be expressed in exponential form as

$$w = \rho(t)\exp[i\phi(t)].$$

Thus

$$f[z(t)] = \rho(t)e^{i\phi(t)} \quad (a \le t \le b); \tag{7.8.3}$$

and, along each of the smooth arcs making up the contour Γ, it follows that (see Exercise 5, Sec. 4.3)

$$f'[z(t)]z'(t) = \frac{d}{dt}f[z(t)] = \frac{d}{dt}[\rho(t)e^{i\phi(t)}] = \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\phi(t)}\phi'(t). \tag{7.8.4}$$

Since $\rho'(t)$ and $\phi'(t)$ are piecewise continuous on the interval $a \le t \le b$, we can now use expressions (7.8.3) and (7.8.4) to write integral (7.8.2) as follows:

$$\int_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{\rho'(t)}{\rho(t)}dt + i\int_a^b \phi'(t)dt = \ln\rho(t)\Big|_a^b + i\phi(t)\Big|_a^b.$$

But

$$\rho(b) = \rho(a) \quad \text{and} \quad \phi(b) - \phi(a) = \Delta_C \arg f(z).$$

Hence

$$\int_C \frac{f'(z)}{f(z)}dz = i\Delta_C \arg f(z). \tag{7.8.5}$$

Another way to evaluate integral (7.8.5) is to use Cauchy's residue theorem. If f has a zero of order m_0 at z_0, then (Sec. 6.7)

$$f(z) = (z - z_0)^{m_0} g(z), \tag{7.8.6}$$

where $g(z)$ is analytic and nonzero at z_0. Hence

$$f'(z_0) = m_0(z - z_0)^{m_0 - 1}g(z) + (z - z_0)^{m_0}g'(z),$$

and so

$$\frac{f'(z)}{f(z)} = \frac{m_0}{z - z_0} + \frac{g'(z)}{g(z)}. \tag{7.8.7}$$

Since $g'(z)/g(z)$ is analytic at z_0, it has a Taylor series representation about that point; and so equation (7.8.7) tells us that $f'(z)/f(z)$ has a simple pole at z_0, with residue m_0. If, on the other hand, f has a pole of order m_p at z_0, we know from the theorem in Sec. 6.5 that

$$f(z) = (z - z_0)^{-m_p}\,\phi(z)\,, \tag{7.8.8}$$

where $\phi(z)$ is analytic and nonzero at z_0. Because expression (7.8.8) has the same form as expression (7.8.6), with the positive integer m_0 in equation (7.8.6) replaced by $-m_p$, it is clear from equation (7.8.7) that $f'(z)/f(z)$ has a simple pole at z_0, with residue $-m_p$. Thus, we observe that the integrand $f'(z)/f(z)$ is analytic inside and on C except at the points inside C at which the zeros and poles of f occur.

Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be the zeros of f of orders $m_1, m_2, \ldots, m_s$, respectively, inside C, and $\beta_1, \beta_2, \ldots, \beta_t$ be the poles of f of orders $n_1, n_2, \ldots, n_t$, respectively, inside C. It follows from the discussion above and the residue theorem that

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}\,dz &= 2\pi i\Big[\sum_{k=1}^{s} \operatorname*{Res}_{z=\alpha_k} \frac{f'(z)}{f(z)} + \sum_{k=1}^{t} \operatorname*{Res}_{z=\beta_k} \frac{f'(z)}{f(z)}\Big] \\ &= 2\pi i\Big[\sum_{k=1}^{s} m_k + \sum_{k=1}^{t} (-n_k)\Big] \\ &= 2\pi i(Z - P).\end{aligned} \tag{7.8.9}$$

Expression (7.8.1) now follows by equating the right-hand sides of equations (7.8.5) and (7.8.9). This completes the proof.

Example. The only singularity of the function $1/z^2$ is a pole of order 2 at the origin, and there are no zeros in the finite plane. In particular, this function is analytic and nonzero on the unit circle

$$z = e^{i\theta}\ (0 \le \theta \le 2\pi).$$

If we let C denote that positively oriented circle, our theorem tells us that

$$\frac{1}{2\pi}\Delta_C \arg\left(\frac{1}{z^2}\right) = -2\,.$$

That is, the image Γ of C under the transformation $w = 1/z^2$ winds around the origin $w = 0$ twice in the clockwise direction. This can be verified directly by noting that Γ has the parametric representation

$$w = e^{-i2\theta}\ (0 \le \theta \le 2\pi).$$