§5.9. Uniqueness of Series Representations

The uniqueness of Taylor and Laurent series representations, anticipated in Secs. 5.3 and 5.5, respectively, follows readily from Theorem 5.8.1 in Sec. 5.8. We consider first the uniqueness of Taylor series representations.

Theorem 5.9.1. *If a series*

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \tag{5.9.1}$$

converges to $f(z)$ *at all points interior to some circle* $|z-z_0| = R$, *then it is the Taylor series expansion for* f.

Proof. To prove this, we write the series representation

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m \quad (|z-z_0| < R). \tag{5.9.2}$$

Then, by Theorem 5.8.1, we may write

$$\int_C g(z) f(z) dz = \sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m dz, \tag{5.9.3}$$

where $g(z)$ is any one of the functions

$$g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z-z_0)^{n+1}} \quad (n = 0,1,2,\ldots) \tag{5.9.4}$$

and C is some circle centered at z_0 and with radius less than R.

By using Theorem 4.13.1, we see

$$\int_C g(z) f(z) dz = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}; \tag{5.9.5}$$

and, since (see Exercise 10, Sec. 4.5)

$$\int_C g(z)(z-z_0)^m dz = \frac{1}{2\pi i} \int_C \frac{dz}{(z-z_0)^{n-m+1}} = \begin{cases} 0, & \text{when } m \neq n, \\ 1, & \text{when } m = n, \end{cases} \tag{5.9.6}$$

it is clear that

$$\sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m dz = a_n. \tag{5.9.7}$$

Because of equations (5.9.5) and (5.9.7), equation (5.9.3) now reduces to

$$\frac{f^{(n)}(z_0)}{n!} = a_n, \quad n = 0,1,2,\ldots$$

and this shows that series (5.9.2) is, in fact, the Taylor series for f about the point z_0.

Note how it follows from Theorem 5.9.1 that if series (5.9.1) converges to zero throughout some neighborhood of z_0, then the coefficients a_n must all be zero.

Our second theorem here concerns the uniqueness of Laurent series representations.

Theorem 5.9.2. *If a series*

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \tag{5.9.8}$$

converges to $f(z)$ *at al points in some annular domain about* z_0, *then it is the Laurent series expansion for* f *in powers of* $z - z_0$ *for that domain.*

Proof. The method of proof here is similar to the one used in proving Theorem 5.9.1. The hypothesis of this theorem tells us that there is an annular domain about z_0 such that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

for each point z in it. Let $g(z)$ be as defined by equation (5.9.4), but now allow n to be a negative integer too. Also, let C be any circle around the annulus, centered at z_0 and taken in the positive sense. Then, using the index of summation m and adapting Theorem 5.8.1 in Sec. 5.8 to series involving both nonnegative and negative powers of $z - z_0$ (Exercise 10), write

$$\int_C g(z) f(z) dz = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m \, dz ,$$

that is,

$$\frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}} = \sum_{m=-\infty}^{\infty} c_m \int_C g(z)(z - z_0)^m \, dz . \qquad (5.9.9)$$

Since equations (5.9.6) are also valid then the integers m and n are allowed to be negative, equation (5.9.9) reduces to

$$\frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}} = c_n ,$$

which is expression (5.4.5), Sec. 5.4, for coefficients in the Laurent series for f in the annulus. This completes the proof.