§6.4. The Three Types of Isolated Singular Points

We saw in Sec. 6.1 that the theory of residues is based on the fact that if f has an isolated singular point z_0, then $f(z)$ can be represented by a Laurent series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_n}{(z-z_0)^n}+\cdots \qquad (6.4.1)$$

in a punctured disk $0<|z-z_0|<R_2$. The portion

$$\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_n}{(z-z_0)^n}+\cdots$$

of the series, involving negative powers of $z-z_0$, is called the *principal part* of f at z_0. We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory that appears in following sections.

Definition 6.4.1. If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is finite, then there exists a positive integer m such that

$$b_m\neq 0 \text{ and } b_{m+1}=b_{m+2}=\cdots=0.$$

In this case, the isolated singular point z_0 is called a *pole of order* m of f. A pole of order $m=1$ of f is usually referred to as a *simple pole* of f.

Thus, the isolated singular point z_0 is a pole of order m of f if and only if the Laurent expansion takes the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_m}{(z-z_0)^m}, \qquad (6.4.2)$$

where $b_m\neq 0$ and $0<|z-z_0|<R_2$.

Example 1. Observe that the function

$$\frac{z^2-2z+3}{z-2}=\frac{z(z-2)+3}{z-2}=z+\frac{3}{z-2}=2+(z-2)+\frac{3}{z-2}(0<|z-2|<\infty)$$

has a simple pole at $z_0=2$. Its residue b_1 there is 3.

Example 2. The function

$$\frac{\sinh z}{z^4}=\frac{1}{z^4}\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\cdots\right)=\frac{1}{z^3}+\frac{1}{3!}\cdot\frac{1}{z}+\frac{z}{5!}+\frac{z^3}{7!}+\cdots(0<|z|<\infty)$$ has a pole of order $m=3$ at $z_0=0$, with residue $b_1=1/6$.

There remain two extremes, the case in which all of the coefficients in the principal part are zero and the one in which an infinite number of them are nonzero.

Definition 6.4.2. When all of the b_n's are zero, the point z_0 is called a *removable singular point* of f.

Thus, the isolated singular point z_0 is a removable singular point of f if and only if the Laurent expansion takes the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots(0<|z-z_0|<R_2). \quad (6.4.3)$$

Note that the residue at a removable singular point is always zero. If we define, or possibly redefine, f at z_0 so that $f(z_0)=a_0$, expansion (6.4.3) becomes valid throughout the entire disk $|z-z_0|<R_2$. Since a power series always represents an analytic function interior to its circle of convergence (Sec. 5.8), it follows that f is analytic at z_0 when it is assigned the

value a_0 there. The singularity at z_0 is, therefore, *removed.*

Example 3. The point $z_0 = 0$ is a removable singular point of the function

$$f(z) = \frac{1-\cos z}{z^2} = \frac{1}{z^2}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)\right]$$

$$= \frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\cdots \quad (0<|z|<\infty).$$

When the value $f(0)=1/2$ is assigned, f becomes an entire function.

Definition 6.4.3. When an infinite number of the coefficients b_n are nonzero, z_0 is said to be an *essential singular point* of f.

Example 4. The point $z_0 = 0$ is an essential singular point of the function

$$f(z) = \sin\frac{1}{z} = \frac{1}{z}-\frac{1}{3!}\frac{1}{z^3}+\frac{1}{5!}\frac{1}{z^5}-\frac{1}{7!}\frac{1}{z^7}+\cdots \quad (0<|z|<\infty).$$

By definition, we obtain a classification of isolated singular points:

$$\text{Isolated singular points}\begin{cases}\text{Removable singular points,}\\ \text{Poles,}\\ \text{Essential singular points.}\end{cases}$$

An important result concerning the behavior of a function near an essential singular point is due to Picard. It states that in each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception, an infinite number of times. More precisely, if z_0 is an essential singular point of a function f, then for every $\delta > 0$, there is at most a point w^* such that for every point $w \in \mathbf{C}\setminus\{w^*\}$, there exist a sequence $\{z_n\}$ of different points in $N(z_0,\delta)$ with $f(z_n) = w(n=1,2,\ldots)$.

Example 5. The function

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{1}{z^n} = 1+\frac{1}{1!}\cdot\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\cdots \quad (0<|z|<\infty)$$

has an essential singular point at $z_0 = 0$, where the residue b_1 is unity. For an illustration of Picard's theorem, let us show that $\exp(1/z)$ assumes the value -1 an infinite number of times in each neighborhood of the origin. To do this, we recall from the example in Sec. 2.18 that

$$\exp z = -1 \text{ when } z = (2n+1)\pi i(n = 0, \pm 1, \pm 2, \ldots).$$

This means that $\exp\frac{1}{z} = -1$ when

$$z = \frac{1}{(2n+1)\pi i}\cdot\frac{i}{i} = -\frac{i}{(2n+1)\pi} \quad (n = 0, \pm 1, \pm 2, \ldots),$$

and an infinite number of these points clearly lie in any given neighborhood of the origin. Since $\exp(1/z) \neq 0$ for any value of z, zero is the exceptional value in Picard's theorem.

In the remaining sections of this chapter, we shall develop in greater depth the theory of the three types of isolated singular points just described. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.