§8.8. Cross Ratios

Lemma 8.5.1. *For every three distinct points* z_2, z_3, z_4 *in the extended plane, there exists a unique fractional linear function* T *such that*

$$T(z_2) = 1, T(z_3) = 0, \text{ and } T(z_4) = \infty. \tag{8.8.1}$$

Proof. For every complex number z, define $T : \mathbf{C}_\infty \to \mathbf{C}_\infty$ by

$$T(z) = \frac{z_2 - z_4}{z_2 - z_3} \cdot \frac{z - z_3}{z - z_4} \text{ if } z_2, z_3, z_4 \in \mathbf{C};\ T(z) = \frac{z - z_3}{z - z_4} \text{ if } z_2 = \infty;$$

$$T(z) = \frac{z_2 - z_4}{z - z_4} \text{ if } z_3 = \infty;\ T(z) = \frac{z - z_3}{z_2 - z_3} \text{ if } z_4 = \infty.$$

Then it is esay to check that T is an FLF satisfying the desired condition. Suppose that S is another FLF satisfying this condition. Then the map

$$M = S^{-1}T : \mathbf{C}_\infty \to \mathbf{C}_\infty$$

is also an FLF such that $M(z_k) = z_k (k = 2,3,4)$. It follows from Exercise 6, Sec. 8.8 that $M(z) = z$ for all z in the plane. This shows that $M = I$, the identity map and so $S = T$. This completes the proof.

Definition 8.8.1. Suppose that z_2, z_3, z_4 be three distinct points in $\mathbf{C}_\infty$ and $T : \boldsymbol{C}_\infty \to \boldsymbol{C}_\infty$ be the unique FLF such that $T(z_2) = 1, T(z_3) = 0$ and $T(z_4) = \infty$. Then for any point $z_1 \in \boldsymbol{C}_\infty$, we say that the value $T(z_1)$ is the *cross ratio* of z_1, z_2, z_3, z_4, denoted by (z_1, z_2, z_3, z_4), that is,

$$T(z_1) = (z_1, z_2, z_3, z_4).$$

From the proof of Lemma 8.8.1, we see that

$$(z_1, z_2, z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3} : \frac{z_1 - z_4}{z_1 - z_3} \text{ if } z_1, z_2, z_3, z_4 \in \mathbf{C}; \tag{8.8.2}$$

and $(\infty, z_2, z_3, z_4) = \dfrac{z_2 - z_4}{z_2 - z_3}$ if $z_2, z_3, z_4 \in \mathbf{C}$ and etc. For example,

$$(z_2, z_2, z_3, z_4) = T(z_2) = 1,\ (z_3, z_2, z_3, z_4) = T(z_3) = 0,\ (z_4, z_2, z_3, z_4) = T(z_4) = \infty.$$

Thus, we have the following consequence.

Corollary 8.8.1. *The unique FLT* $T : \boldsymbol{C}_\infty \to \boldsymbol{C}_\infty$ *such that*

$$T(z_2) = 1, T(z_3) = 0 \text{ and } T(z_4) = \infty$$

is given by $T(z) = (z, z_2, z_3, z_4)$ *for all* $z \in \boldsymbol{C}_\infty$. *Also,* $(z, 1, 0, \infty) = z$ *for all* z *in the plane.*

Corollary 8.8.2. *If* z_2, z_3, z_4 *are distinct points in the extended plane, and* $T : \boldsymbol{C}_\infty \to \boldsymbol{C}_\infty$ *is an FLF, then*

$$(T(z_1), T(z_2), T(z_3), T(z_4)) = (z_1, z_2, z_3, z_4) \tag{8.8.3}$$

for any point $z_1 \in \boldsymbol{C}_\infty$.

Proof. Let $S(z) = (z, z_2, z_3, z_4)$ and define $M = ST^{-1}$, then M is an FLF such that

$$M(T(z_2)) = S(z_2) = 1,\quad M(T(z_3)) = S(z_3) = 0,\ M(T(z_4)) = S(z_4) = \infty.$$

Thus, $M(z) = (z, T(z_2), T(z_3), T(z_4))$ for all $z \in \boldsymbol{C}_\infty$. Especially,

$$(T(z_1), T(z_2), T(z_3), T(z_4)) = M(T(z_1)) = S(z_1) = (z_1, z_2, z_3, z_4).$$

This completes the proof.

Theorem 8.8.1. *There is always a fractional linear transformation that maps three given* *distinct points* z_1, z_2*, and* z_3 *in the extended* z*-plane onto three specified distinct points* w_1, w_2*, and* w_3*, in the extended* w*-plane, respectively.*

Proof. Define two FLTs S, T by

$$S(z) = (z, z_2, z_3, z_4),\ T(w) = (w, w_2, w_3, w_4), \tag{8.8.4}$$

then from Corollary 8.8.1 we know that $M = T^{-1}S$ is an FLF such that $M(z_k) = w_k$ for $k = 1,2,3$. To prove the uniqueness of the map, let T and S be two such linear fractional transformations. Then, $S^{-1}[T(z_k)] = z_k (k = 1,2,3)$, the results in Exercises 6 shows that $S^{-1}[T(z)] = z$ for all z. Thus show that $T(z) = S(z)$ for all z. The proof is completed.

From Corollary 8.8.2, the unique FLF F which maps distinct points z_1, z_2, z_3 onto distinct points w_1, w_2, w_3 must satisty the following equation

$$(z_2, z, z_1, z_3) = (w_2, w, w_1, w_3),$$

where $w = F(z)$ provided $z \neq z_1, z \neq z_3$. In the case that all specified points are all finite, the above eqution becomes

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \tag{8.8.5}$$

where $w = F(z)$ for all $z \in C_\infty$.

Example 1. The transformation found in Example 1, Sec.8.7, required that

$$z_1 = -1, z_2 = 0, z_3 = 1 \text{ and } w_1 = -i, w_2 = 1, w_3 = i.$$

Using equation (8.8.5) to write

$$\frac{(w + i)(1 - i)}{(w - i)(1 + i)} = \frac{(z + 1)(0 - 1)}{(z - 1)(0 + 1)}$$

and then solving for w in terms of z, we arrive at the transformation

$$w = \frac{i - z}{i + z},$$

found earlier.

If equation (8.8.5) is modified properly, it can also be used when the point at infinity is one of the prescribed points in either the (extended) z or w plane. Suppose, for instance, that $z_1 = \infty$. Since any fractional linear transformation is continuous on the extended plane, we need only replace z_1 on the right-hand side of equation (8.8.5) by $1/z_1$, clear fractions, and let z_1 tend to zero:

$$\lim_{z_1 \to 0} \frac{(z - 1/z_1)(z_2 - z_3)}{(z - z_3)(z_2 - 1/z_1)} \cdot \frac{z_1}{z_1} = \lim_{z_1 \to 0} \frac{(z_1 z - 1)(z_2 - z_3)}{(z - z_3)(z_1 z_2 - 1)} = \frac{z_2 - z_3}{z - z_3}.$$

The desired modification of equation (8.5.5) is, then,

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{z_2 - z_3}{z - z_3}.$$

Note that this modification is obtained formally by simply deleting the factors involving z_1 in equation (8.5.5). It is easy to check that the same formal approach applies when any of the other prescribed points is ∞.

Example 2. In Example 2, Sec.8.7, the prescribed points were

$$z_1 = 1, z_2 = 0, z_3 = -1 \text{ and } w_1 = i, w_2 = \infty, w_3 = 1.$$

Find the desired FLF.

Solution. In this case, we use the modification

$$\frac{w - w_1}{w - w_3} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

of equation (8.5.5), which tells us that

$$\frac{w-i}{w-1}=\frac{(z-1)(0+1)}{(z+1)(0-1)}.$$

Solving here for w, we arrive at the desired transformation:

$$w=\frac{(i+1)z+(i-1)}{2z}.$$