§3.5. Complex Power Functions

1.Definition of a complex power function

When c is any complex number, the complex power z^c of a nonzero complex number z is defined by means of the equation

$$z^c = e^{c\mathrm{Log}z}, \quad z \neq 0. \tag{3.5.1}$$

Thus, we obtain a multiple-valued function $w = z^c (z \neq 0)$, called a *complex power function*.

2. Examples

Example 1. Powers of z are, in general, multiple-valued, as illustrated by writing

$$i^{-2i} = \exp(-2i\mathrm{Log}i)$$

and then

$$\mathrm{Log}i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(2n + \frac{1}{2}\right)\pi (n = 0, \pm 1, \pm 2, \ldots).$$

This shows that

$$i^{-2i} = \exp[(4n+1)\pi](n = 0, \pm 1, \pm 2, \ldots). \tag{3.5.2}$$

Note that these values of i^{-2i} are *all* real numbers.

Since the exponential function has the property $1/e^z = e^{-z}$, one can see that

$$\frac{1}{z^c} = \frac{1}{\exp(c\mathrm{Log}z)} = \exp(-c\mathrm{Log}z) = z^{-c}$$

and, in particular, that $1/i^{2i} = i^{-2i}$. According to expression (3.5.2), then,

$$\frac{1}{i^{2i}} = \exp[(4n+1)\pi](n = 0, \pm 1, \pm 2, \ldots). \tag{3.5.3}$$

The *principal value* of z^c occurs when $\mathrm{Log}z$ is replaced by $\log z$ in definition (3.5.1):

$$\text{P.V. } z^c = e^{c \log z} = (z^c)_{-\pi}. \tag{3.5.5}$$

Example 2. The principal value of $(-i)^i$ is

$$\exp[i\log(-i)] = \exp\left[i\left(\ln 1 - i\frac{\pi}{2}\right)\right] = \exp\frac{\pi}{2}.$$

That is,

$$\text{P.V. } (-i)^i = \exp\frac{\pi}{2}. \tag{3.5.6}$$

Example 3. The principal branch of $z^{2/3}$ can be written

$$\exp\left(\frac{2}{3}\log z\right) = \exp\left(\frac{2}{3}\ln r + \frac{2}{3}i\arg z\right) = \sqrt[3]{r^2}\exp\left(i\frac{2\arg z}{3}\right).$$

Thus

$$\text{P.V. } z^{2/3} = \sqrt[3]{r^2}\cos\frac{2\arg z}{3} + i\sqrt[3]{r^2}\sin\frac{2\arg z}{3}. \tag{3.5.7}$$

This function is analytic in the domain $D_{-\pi}$, as one can see directly from Theorem 2.12.1.