§7.5. Indented Paths

In this and the following section, we illustrate the use of *indented* paths. We begin with an important limit that will be used in the example in this section.

Theorem 7.5.1. *Suppose that*

(i) *a function* $f(z)$ *has a simple pole at a point* $z = x_0$ *on the real axis, with a Laurent series representation in a punctured disk* $0 <| z - x_0 |< R_2$ (Fig. 7-7) *and with residue* B_0;

(ii) C_ρ *denotes the upper half of a circle* $| z - z_0 |= \rho$, *where* $\rho < R_2$ *and the clockwise direction is taken.*

Then

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz = -B_0\pi i. \tag{7.5.1}$$

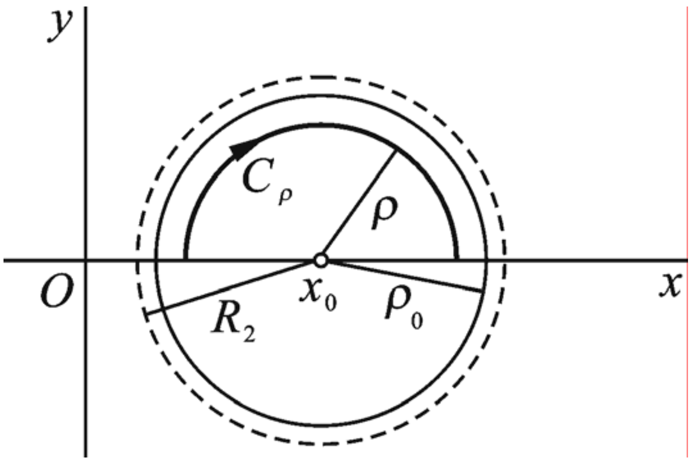

Fig. 7-7

Proof. From the assumption (i), the La … nction f can be written as

$$f(z) = g(z) + \frac{B_0}{z - x_0} \quad (0 <| z - x_0 |< R_2),$$

where $g(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n \quad (| z - x_0 |< R_2)$. Thus

$$\int_{C_\rho} f(z)dz = \int_{C_\rho} g(z)dz + B_0\int_{C_\rho}\frac{dz}{z - x_0}. \tag{7.5.2}$$

Now the function $g(z)$ is continuous when $| z - z_0 |< R_2$, according to Corollary 5.8.1. Hence if we choose a number ρ_0 such that $\rho < \rho_0 < R_2$ (see Fig. 7-7), it must be bounded on the *closed* disk $| z - x_0 |\le \rho_0$, according to Sec. 2.7. That is, there is a nonnegative constant M such that $| g(z) |\le M$ whenever $| z - x_0 |\le \rho_0$; and, since the length L of the path C_ρ is $L = \pi\rho$, it follows that

$$\left|\int_{C_\rho} g(z)dz\right| \le ML = M\pi\rho.$$

Consequently,

$$\lim_{\rho\to 0}\int_{C_\rho} g(z)dz = 0. \tag{7.5.3}$$

Since the semicircle $-C_\rho$ has parametric representation

$$z = x_0 + \rho e^{i\theta} \quad (0 \le \theta \le \pi),$$

the second integral on the right in equation (7.5.2) has the value

$$\int_{C_\rho} \frac{dz}{z - x_0} = -\int_{-C_\rho} \frac{dz}{z - x_0} = -\int_0^\pi \frac{1}{\rho e^{i\theta}} \rho i e^{i\theta} d\theta = -i\int_0^\pi d\theta = -i\pi .$$

Thus

$$\lim_{\rho\to 0} \int_{C_\rho} \frac{dz}{z - x_0} = -i\pi . \tag{7.5.4}$$

Limit (7.5.1) now follows by letting ρ tend to zero on each side of equation (7.5.2) and referring to limits (7.5.3) and (7.5.4). This completes the proof.

Example. Modifying the method used in Secs. 7.3 and 7.4, we derive here the integration formula

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \tag{7.5.5}$$

by integrating e^{iz}/z around the simple closed contour shown in Fig. 7-8. In that figure, ρ and R denote positive real numbers, where $\rho < R$; and L_1 and L_2 represent the intervals $\rho \le x \le R$ and $-R \le x \le -\rho$, respectively, on the real axis. While the semicircle C_R is as in Secs. 7.3 and 7.4, the semicircle C_ρ is introduced here in order to avoid integrating through the singularity $z = 0$ of e^{iz}/z.

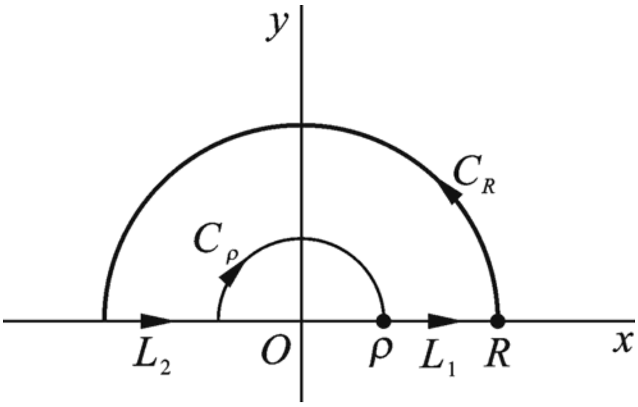

Fig. 7-8

The Cauchy integral theorem te

$$\int_{L_1} \frac{e^{iz}}{z} dz + \int_{C_R} \frac{e^{iz}}{z} dz + \int_{L_2} \frac{e^{iz}}{z} dz + \int_{C_\rho} \frac{e^{iz}}{z} dz = 0 .$$

That is,

$$\int_{L_1} \frac{e^{iz}}{z} dz + \int_{L_2} \frac{e^{iz}}{dz} = -\int_{C_\rho} \frac{e^{iz}}{z} dz - \int_{C_R} \frac{e^{iz}}{z} dz . \tag{7.5.6}$$

Moreover, since the legs L_1 and L_2 have parametric representations

$$z = x(\rho \le x \le R) \quad \text{and} \quad z = x(-R \le x \le -\rho), \tag{7.5.7}$$

respectively, the left-hand side of equation (7.5.6) can be written

$$\int_{L_1} \frac{e^{iz}}{z} dz + \int_{L_2} \frac{e^{iz}}{z} dz = \int_\rho^R \frac{e^{ix}}{x} dx + \int_{-R}^{-\rho} \frac{e^{ix}}{x} dx = 2i\int_\rho^R \frac{\sin x}{x} dx .$$

Consequently,

$$2i\int_\rho^R \frac{\sin x}{x} dx = -\int_{C_\rho} \frac{e^{iz}}{z} dz - \int_{C_R} \frac{e^{iz}}{z} dz . \tag{7.5.8}$$

Now, from the Laurent series representation

$$\frac{e^{iz}}{z} = \frac{1}{z}\left[1 + \frac{(iz)}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots\right]$$

$$= \frac{1}{z} + \frac{i}{1!} + \frac{i^2}{2!}z + \frac{i^3}{3!}z^2 + \cdots \quad (0 < |z| < \infty)$$

it is clear that e^{iz}/z has a simple pole at the origin, with residue unity. So, according to Theorem 7.5.1,

$$\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iz}}{z}\,dz = -\pi i\,.$$

Also, since

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{R}$$

when z is a point on C_R, we know from Jordan's lemma (Theorem 7.4.1) that

$$\lim_{R\to 0}\int_{C_R} \frac{e^{iz}}{z}\,dz = 0\,.$$

Thus, by letting ρ tend to 0 in equation (7.5.8) and then letting R tend to ∞, we arrive at the result $2i\int_0^\infty \frac{\sin r}{r}\,dr = \pi i$, which is, in fact, formula (7.5.5).