§1.3. Further Properties

In this section, we mention a number of other algebraic properties of addition and multiplication of complex numbers that follow from the ones already described in Sec.1.2. Because such properties continue to be anticipated, the reader can easily pass to Sec.1.4 without serious disruption.

1. Expunctive law

If $z_1 z_2 = 0$, then either $z_1 = 0$ or $z_2 = 0$; or possibly both z_1 and z_2 equal zero. Another way to state this result is that *if two complex numbers* z_1 *and* z_2 *are nonzero, then so is their product* $z_1 z_2$.

2. Division

Division by a nonzero complex number is defined as follows:

$$\frac{z_1}{z_2} = z_1 z_2^{-1} \quad (z_2 \neq 0) \tag{1.3.1}$$

If $z_1 = (x_1, y_1) = x_1 + iy_1$ and $z_2 = (x_2, y_2) = x_2 + iy_2$, then

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \quad (z_2 \neq 0). \tag{1.3.2}$$

Although expression (1.3.2) is not easy to remember, it can be obtained by writing (see Exercise 4)

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}. \tag{1.3.3}$$

3. Useful identities

$$\frac{1}{z_2} = z_2^{-1} \quad (z_2 \neq 0). \tag{1.3.4}$$

$$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \quad (z_2 \neq 0). \tag{1.3.5}$$

$$(z_1 z_2)(z_1^{-1} z_2^{-1}) = (z_1 z_1^{-1})(z_2 z_2^{-1}) = 1 \quad (z_2 \neq 0).$$

$$\frac{1}{z_1 z_2} = (z_1 z_2)^{-1} = z_1^{-1} z_2^{-1} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) \qquad (z_1 \neq 0, z_2 \neq 0). \tag{1.3.6}$$

$$\frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) \quad (z_3 \neq 0, z_4 \neq 0). \tag{1.3.7}$$

Example. Computations such as the following are now justified:

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right) = \frac{1}{(2-3i)(1+i)} = \frac{1}{5-i}\cdot\frac{5+i}{5+i} = \frac{5+i}{(5-i)(5+i)}$$

$$= \frac{5+i}{26} = \frac{5}{26} + \frac{i}{26} = \frac{5}{26} + \frac{1}{26}i.$$

4. Binomial formula

If z_1 and z_2 are any two complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{n-k} z_2^k \quad (n = 1, 2, \ldots)\ (\textit{Binomial Formula}) \tag{1.3.8}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $(k = 0, 1, 2, \ldots, n)$ and where it is agreed that $0! = 1$.