§5.7. Continuity of Sums of Power Series

Our next theorem is an important consequence of uniform convergence, discussed in the previous section.

Theorem 5.7.1. *The sum function* $S(z)$ *of a power series*

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n \tag{5.7.1}$$

is continuous at each point inside its disk of convergence $|z - z_0| < R$.

Proof. It suffices to prove that if z_1 is a point inside that circle, then for each positive number ε, there is positive number δ such that

$$|S(z) - S(z_1)| < \varepsilon \quad \text{whenever} \quad |z - z_1| < \delta. \tag{5.7.2}$$

To show this, let z_1 be any point inside the disk of convergence of (5.7.1), then there is a $r > 0$ such that $N(z_1, r) \subset N(z_0, R)$, see Fig. 5-8. Let $S_n(z)$ denote the sum of the first n terms of series (5.7.1) and write the remainder function

$$\rho_n(z) = S(z) - S_n(z) \qquad (|z - z_0| < R).$$

Then, because

$$S(z) = S_n(z) + \rho_n(z) \qquad (|z - z_0| < R),$$

one can see that

$$|S(z) - S(z_1)| \le |S_n(z) - S_n(z_1)| + |\rho_n(z)| + |\rho_n(z_1)|. \tag{5.7.3}$$

Take a closed disk $|z - z_0| \le R_0$ such that $N(z_1, r) \subset N(z_0, R_0) \subset N(z_0, R)$. Then the uniform convergence stated in Theorem 5.6.2, Sec. 5.6, ensures that there is a positive integer N such that

$$|\rho_n(z)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad n > N \quad \text{and} \quad |z - z_0| < R_0. \tag{5.7.4}$$

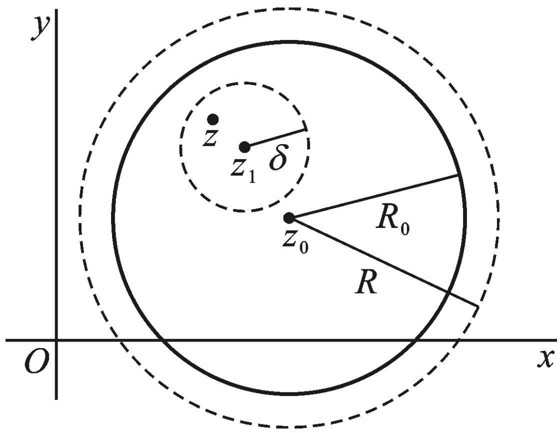

Fig. 5-8

Now the partial sum $S_{N+1}(z)$ is a polynomial and is, therefore, continuous at z_1, we can find a $\delta > 0$ with $\delta < r$ such that

$$|S_{N+1}(z) - S_{N+1}(z_1)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta. \tag{5.7.5}$$

By writing $n = N + 1$ in inequality (5.7.3) and using (5.7.4) and (5.7.5), we now find that

$$\left|S(z)-S(z_1)\right| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \quad \text{whenever} \quad \left|z-z_1\right|<\delta .$$

This is statement (5.7.2), and the theorem is now established.

By writing $w = 1/(z - z_0)$, one can modify the two theorems in the previous section and the theorem here so as to apply to series of the type

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} . \tag{5.7.6}$$

If, for instance, series (5.7.6) converges at a point $z_1 (z_1 \neq z_0)$, then the series

$$\sum_{n=1}^{\infty} b_n w^n$$

must converge absolutely to a continuous function when

$$|w| < \frac{1}{|z_1 - z_0|} . \tag{5.7.7}$$

Thus, since inequality (5.7.7) is the same as $|z - z_0| > |z_1 - z_0|$, series (5.7.6) must converge absolutely to a continuous function in the domain *exterior* to the circle $|z - z_0| = R_1$, where $R_1 = |z_1 - z_0|$. Also, we know that if a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

is valid in an annulus $R_1 < |z - z_0| < R_2$, then both of the series on the right converge uniformly in any closed annulus which is concentric to and interior to that region of validity. We conclude this discussion as follows.

Theorem 5.7.2. *If a Laurent series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \tag{5.7.8}$$

is convergent on an annulus

$$A(z_0, R_1, R_2) = \{z : R_1 < |z - z_0| < R_2\} , \tag{5.7.9}$$

then the two series in (5.7.8) *are uniformly convergent on any bounded closed set* D *contained in the annulus and the sum function* $f(z)$ *is continuous on the annulus.*