§6.10. Behavior of f Near Isolated Singular Points

As already indicated in Sec. 6.4, if a function f has an isolated singular point at z_0, then z_0 is a pole, a removable singular point, or an essential singular point. In this section, we develop the differences in behavior somewhat further.

Theorem 6.10.1. *Suppose that* z_0 *is an singular point of a function* f, *then* z_0 *is a removable singular point of* f *if and only if* f *is bounded on some deleted neighborhood* $0 < |z - z_0| < \varepsilon$ *of a point* z_0.

Proof. *Sufficiency.* Since the point z_0 is an isolated singular point of f, $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (6.10.1)$$

throughout a deleted neighborhood $0 < |z - z_0| < R$. If C denotes a positively oriented circle $|z - z_0| = \rho$, where $\rho < \min\{\varepsilon, R\}$ (Fig. 6-11), we know from Sec. 5.4 that the coefficients b_n in expansion (6.10.1) can be written

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \, (n = 0,1,2,\ldots). \qquad (6.10.2)$$

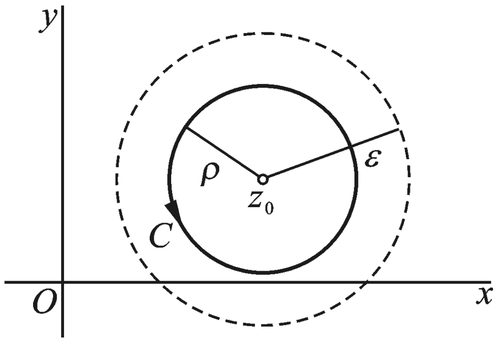

Fig. 6-11

Now the boundedness on f tells us that there is a positive constant M such that $|f(z)| \le M$ whenever $0 < |z - z_0| < \varepsilon$. Hence it follows from expression (6.10.2) that

$$|b_n| \le \frac{1}{2\pi} \cdot \frac{M}{\rho^{-n+1}} 2\pi\rho = M\rho^n \quad (n = 1, 2, \ldots).$$

Since the coefficients b_n are constants and since ρ can be chosen arbitrarily small, we may conclude that $b_n = 0 (n = 1, 2, \ldots)$. This tells us that z_0 is a removable singular point of f.

Necessity. Suppose that z_0 is a removable singular point of f. Then by definition, $f(z)$ has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (0 < |z - z_0| < R). \qquad (6.10.3)$$

Since the power series in the right hand side of (6.10.3) is convergent for all z with $0 < |z - z_0| < R$, the sum of that series is continuous at z_0. So, $\lim_{z \to z_0} f(z)$ exists and equals to a_0. Hence, f is bounded on some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of a point z_0. This completes the proof.

Corollary 6.10.1. *Suppose that* z_0 *is an singular point of a function* f, *then* z_0 *is a* *removable singular point of* f *if and only if* $\lim_{z\to z_0} f(z)$ *exists in the complex plane.*

The next theorem emphasizes how the behavior of f near the behavior near a pole is fundamentally different from a removable singular point.

Theorem 6.10.2. *If* z_0 *is an isolated singular point of a function* f*, then it is a pole of a function* f *if and only if* $\lim_{z\to z_0} f(z)=\infty$.

Proof. *Necessity*. Let z_0 be a pole of f. To verify $\lim_{z\to z_0} f(z)=\infty$, we assume that f has a pole of order m at z_0. Using Theorem 6.5.1 tells us that $f(z)=\dfrac{\phi(z)}{(z-z_0)^m}$, where $\phi(z)$ is analytic and nonzero at z_0. Since

$$\lim_{z\to z_0}\frac{1}{f(z)}=\lim_{z\to z_0}\frac{(z-z_0)^m}{\phi(z)}=\frac{\lim_{z\to z_0}(z-z_0)^m}{\lim_{z\to z_0}\phi(z)}=\frac{0}{\phi(z_0)}=0,$$

limit (6.10.4) holds, according to the theorem in Sec. 2.6 regarding limits that involve the point at infinity.

Sufficiency. Suppose that $\lim_{z\to z_0} f(z)=\infty$. Then the function $F(z)=\dfrac{1}{f(z)}$ has an isolated singular point at z_0 and $\lim_{z\to z_0} F(z)=0$. Thus, we see from Theorem 6.10.1 that $F(z)$ has the Laurent expansion

$$F(z)=c_0+c_1(z-z_0)+\cdots \quad (0<|z-z_0|<\varepsilon).$$

Since $\lim_{z\to z_0} F(z)=0$, $c_0=0$. But $F(z)$ is not zero in $0<|z-z_0|<\varepsilon$. Hence, there exists a positive integer m such that

$$c_k=0(0\le k<m), c_m\ne 0.$$

Thus, $F(z)=c_m(z-z_0)^m+c_{m+1}(z-z_0)^{m+1}+\cdots \quad (0<|z-z_0|<\varepsilon)$. Put

$$\varphi(z)=c_m+c_{m+1}(z-z_0)+\cdots,$$

then φ is analytic and nonzero at z_0, and

$$f(z)=\frac{1}{F(z)}=\frac{1/\varphi(z)}{(z-z_0)^m} \quad (0<|z-z_0|<\varepsilon).$$

It follows from Theorem 6.5.1 that z_0 is a pole of the function f. This completes the proof.

Form Corolary 6.10.1 and Theorem 6.10.2, we get

Corollary 6.10.2. *If* z_0 *is an isolated singular point of a function* f*, then it is an essential singularity of* f *if and only if the limit* $\lim_{z\to z_0} f(z)$ *does exist in the extended complex plane.*

Theorem 6.10.3(Casorati-Weierstrass). *Suppose that* z_0 *is an essential singularity of a function* f*, and let* w_0 *be any complex number. Then, for any positive numbers* ε *and* δ*, the inequality* $|f(z)-w_0|<\varepsilon$ *holds at some point* z *with* $0<|z-z_0|<\delta$ *of* z_0(Fig. 6-12).

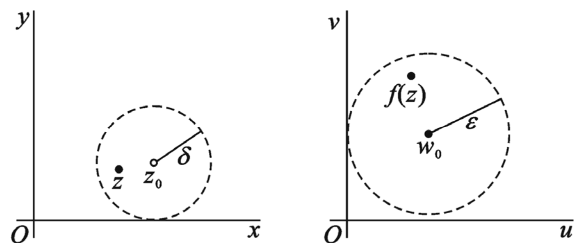

Fig. 6-12

Proof. The proof is by contradiction. Since z_0 is an isolated singularity of f, there is a deleted neighborhood $0 < |z - z_0| < \delta$ throughout which f is analytic; and we assume that condition (6.10.4) is not satisfied for any point z there. Thus $|f(z) - w_0| \geq \varepsilon$ when $0 < |z - z_0| < \delta$; and so the function

$$g(z) = \frac{1}{f(z) - w_0} \quad (0 < |z - z_0| < \delta) \tag{6.10.5}$$

is bounded and analytic in its domain of definition. Hence, According to Theorem 6.10.1, z_0 is a removable singularity of g; and we let g be defined at z_0 so that it is analytic there.

If $g(z_0) \neq 0$, the function $f(z)$, which can be written

$$f(z) = \frac{1}{g(z)} + w_0 \tag{6.10.6}$$

when $0 < |z - z_0| < \delta$, becomes analytic at z_0 if it is defined there as

$$f(z_0) = \frac{1}{g(z_0)} + w_0 .$$

But this means that z_0 is a removable singularity of f, not an essential one, and we have a contradiction.

If $g(z_0) = 0$, the function g must have a zero of some finite order m (Sec. 6.7) at z_0 because $g(z)$ is not identically equal to zero in the neighborhood $|z - z_0| < \delta$. In view of equation (6.10.6), then, f has a pole of order m at z_0 (see Theorem 6.9.1 in Sec. 6.9). So, once again, we have a contradiction; and Theorem 6.10.3 here is proved.

Theorem 6.10.3 says indeed that the values of a function f on any deleted neighborhood $N^\circ(z_0, \delta)$ of an essential singular point z_0 of it are dense in whole of the plane, that is,

$$\overline{f(N^\circ(z_0, \delta))} = \mathbf{C} \quad \text{for all } \delta > 0 .$$