§3.8. Inverse Trigonometric and Hyperbolic Functions

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms. In order to define the inverse sine function $\sin^{-1}$, we write

$$w = \sin^{-1} z$$

when $z = \sin w$. That is, $w = \sin^{-1} z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i}.$$

If we put this equation in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0,$$

which is quadratic in e^{iw}, and solve for e^{iw} [see Exercise 8 (a), Sec. 1.9], we find that

$$e^{iw} = iz + (1 - z^2)^{1/2}, \tag{3.8.1}$$

where $(1 - z^2)^{1/2}$ is, of course, a double-valued function of z. Taking logarithms of each side of equation (3.8.1) and recalling that $w = \sin^{-1} z$, we arrive at the expression

$$\sin^{-1} z = -i\text{Log}[iz + (1 - z^2)^{1/2}]. \tag{3.8.2}$$

The following example illustrates the fact that $\sin^{-1} z$ is a multiple-valued function, with infinitely many values at each point z.

Example. Expression (3.8.2) tells us that

$$\sin^{-1}(-i) = -\text{Log}(1 \pm \sqrt{2}).$$

But

$$\text{Log}(1 + \sqrt{2}) = \ln(1 + \sqrt{2}) + 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

and

$$\text{Log}(1 - \sqrt{2}) = \ln(\sqrt{2} - 1) + (2n + 1)\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Since $\ln(\sqrt{2} - 1) = \ln\frac{1}{1 + \sqrt{2}} = -\ln(1 + \sqrt{2})$,

Then, the numbers

$$(-1)^n \ln(1 + \sqrt{2}) + n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

constitute the set of values of $\text{Log}(1 \pm \sqrt{2})$. Thus, in rectangular form,

$$\sin^{-1}(-i) = n\pi + i(-1)^{n+1}\ln(1 + \sqrt{2}) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

One can apply the technique used to derive expression (3.8.2) for $\sin^{-1} z$ to show that

$$\cos^{-1} z = -i\text{Log}[z + i(1 - z^2)^{1/2}] \tag{3.8.3}$$

and that

$$\tan^{-1} z = \frac{i}{2}\text{Log}\frac{i + z}{i - z}. \tag{3.8.4}$$

The functions $\cos^{-1} z$ and $\tan^{-1} z$ are also multiple-values. When specific branches of the square root and logarithmic functions are used, all three inverse functions become single-valued and analytic because they are then compositions of analytic functions.

The derivatives of these three functions are readily obtained from the above expressions. The derivatives of the first two depend on the values chosen for the square roots:

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1 - z^2)^{1/2}}, \quad \frac{d}{dz}\cos^{-1} z = \frac{-1}{(1 - z^2)^{1/2}}.$$

The derivative of the last one $\frac{d}{dz}\tan^{-1} z = \frac{1}{1 + z^2}$, does not, however, depend on the manner in which the function is made single-valued.

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

$$\sinh^{-1} z = \mathrm{Log}[z + (z^2 + 1)^{1/2}], \quad \cosh^{-1} z = \mathrm{Log}[z + (z^2 - 1)^{1/2}]$$

and

$$\tanh^{-1} z = \frac{1}{2} \mathrm{Log} \frac{1 + z}{1 - z} .$$

Finally, we remark that common alternative notation for all of these inverse functions is $\arcsin z$, etc.