§4.2. Definite Integrals of Functions w

Let w be a complex-valued function of a real variable t in $[a,b]$, then it can be written as

$$w(t) = u(t) + iv(t), \quad \forall t \in [a,b], \tag{4.2.1}$$

where u and v are real-valued.

Definition 4.2.1. For a function w as in (4.2.1), if u and v are Riemann integrable over $[a,b]$, then we say that w is integrable on $[a,b]$ and the definite integral of w over $[a,b]$ is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt. \tag{4.2.2}$$

Thus

$$\mathrm{Re}\int_a^b w(t)dt = \int_a^b \mathrm{Re}[w(t)]dt \quad \text{and} \quad \mathrm{Im}\int_a^b w(t)dt = \int_a^b \mathrm{Im}[w(t)]dt. \tag{4.2.3}$$

Example 1. For an illustration of definition (4.1.2), we compute

$$\int_0^1 (1+it)^2 dt = \int_0^1 [(1-t^2) + i2t]dt = \int_0^1 (1-t^2)dt + i\int_0^1 2tdt = \frac{2}{3} + i.$$

Improper integrals of w over unbounded intervals are defined in a similar way.

The existence of the integrals of u and v in definition (4.2.2) is ensured if those functions are *piecewise continuous* on the interval $[a,b]$. Such a function is continuous everywhere in the stated interval except possibly for a finite number of points where, although discontinuous, it has one-sided limits. Of course, only the right-hand limit is required at a; and only the left-hand limit is required at b. When both u and v are piecewise continuous, the function w is said to be *piecewise continuous*. Thus, every piecewise continuous complex-valued function on the interval $[a,b]$ is integrable over the interval.

Some basic properties of the integrals defined here are listed in the following theorem.

Theorem 4.2.1. *Suppose that the complex-valued functions* w, w_1, w_2 *are all integrable over the interval* $[a,b]$, *then*

(1) *The function* $w_1 + w_2$ *is integrable over* $[a,b]$ *and*

$$\int_a^b (w_1(t) + w_2(t))dt = \int_a^b w_1(t)dt + \int_a^b w_2(t)dt; \tag{4.2.4}$$

(2) *For every complex number* c, *the function* cw *is integrable over* $[a,b]$ *and*

$$\int_a^b cw(t)dt = c\int_a^b w(t)dt;$$

(3) *When* $a < c < b$, w *is integrable over* $[a,c]$ *and* $[c,b]$, *and*

$$\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt;$$

(4) *The function* $|w|$ *is integrable over* $[a,b]$ *and*

$$\left|\int_a^b w(t)dt\right| \le \int_a^b |w(t)|\, dt. \tag{4.2.5}$$

Proof. The proofs for (1) to (3) are easy to do by recalling corresponding results in calculus. Next, we give the proof for (4). By Definition 4.2.1, we know that the real and imaginary parts u and v of w are all Riemann integrable over $[a,b]$, and so the function

$$|w(t)| = \sqrt{(u(t))^2 + (v(t))^2}$$

is Riemann integrable over $[a,b]$. To show the inequality (4.2.4) is valid, we may assume that the integral on the left is a *nonzero* complex number. If r_0 is the modulus and θ_0 is an argument of the integral on the left, then

$$\int_a^b w(t)dt = r_0 e^{i\theta_0}.$$

Solving for r_0, we write

$$r_0 = \int_a^b e^{-i\theta_0} w(t)dt. \tag{4.2.6}$$

Now the left-hand side of this equation is a real number, and so the right-hand side is real, too. Thus, using the fact that the real part of a real number is the number itself and referring to the first of properties (4.2.3), we see that the right-hand side of equation (4.2.6) can be rewritten in the following way:

$$r_0 = \int_a^b e^{-i\theta_0} w(t)dt = \mathrm{Re}\int_a^b e^{-i\theta_0} w(t)dt = \int_a^b \mathrm{Re}(e^{-i\theta_0} w(t))dt. \tag{4.2.7}$$

But

$$\mathrm{Re}(e^{-i\theta_0} w(t)) \le |e^{-i\theta_0} w(t)| = |e^{-i\theta_0}|\,|w(t)| = |w(t)|, \quad \forall t \in [a,b];$$

and so, according to equation (4.2.7), we have

$$r_0 \le \int_a^b |w(t)|\, dt.$$

The proof is completed.

The fundamental theorem of calculus, involving antiderivatives (i.e., primitive functions), can be extended so as to apply to integrals of the type (4.2.2).

Theorem 4.2.2. *Suppose that the functions*

$$w(t) = u(t) + iv(t) \quad \text{and} \quad W(t) = U(t) + iV(t)$$

are continuous on the interval $[a,b]$ *and* $W'(t) = w(t)$ *when* $t \in [a,b]$, *then*

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)\Big|_a^b.$$

Proof. Since $W'(t) = w(t)$, we have $U'(t) = u(t)$ and $V'(t) = v(t)$. Hence, from the fundamental theorem of calculus and Definition 4.2.1, we obtain that

$$\begin{aligned}\int_a^b w(t)dt &= \int_a^b u(t)dt + i\int_a^b v(t)dt \\ &= U(t)\Big|_a^b + iV(t)\Big|_a^b \\ &= [U(b) + iV(b)] - [U(a) + iV(a)].\end{aligned}$$

Thus, we deduce that

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)\Big|_a^b.$$

This completes the proof.

Example 2. Since $(e^{it})' = ie^{it}$ (See Sec. 4.1), we have $e^{it} = (-ie^{it})'$ and so

$$\begin{aligned}\int_0^{\pi/4} e^{it}dt &= -ie^{it}\Big|_0^{\pi/4} = -ie^{i\pi/4} + i \\ &= -i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + i = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).\end{aligned}$$