§3.4. Some Identities on Logarithms

As suggested by relations (3.2.3) in Sec. 3.2, as well as Exercises 3, 4, and 5 with Sec. 3.3, some identities involving logarithms in calculus carry over to complex analysis and others do not. In this section, we derive a few that do carry over, sometimes with qualifications as to how they are to be interpreted. A reader who wishes to pass to Sec. 3.2 can simply refer to results here when needed.

1.Operations of $\mathrm{Log} z$

If z_1 and z_2 denote any two nonzero complex numbers, then

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log} z_1 + \mathrm{Log} z_2, \quad \forall z_1 z_2 \neq 0. \tag{3.4.2}$$

$$\mathrm{Log}\frac{z_1}{z_2} = \mathrm{Log} z_1 - \mathrm{Log} z_2, \tag{3.4.3}$$

2. Properties of $\mathrm{Log} z$

We include two other properties of $\mathrm{Log} z$ that will be of special interest in

$$z^n = e^{n\mathrm{Log} z} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{3.4.4}$$

When $n = 1$, this reduces, of course, to relation (3.2.3), Sec. 3.2. Equation (3.4.4) is readily verified by writing $z = re^{i\theta}$ and noting that each side becomes $r^n e^{in\theta}$. Also,

$$z^{1/n} = \exp(\frac{1}{n}\mathrm{Log} z) \quad (n = 1, 2, \ldots) \tag{3.4.5}$$

That is, the term on the right here has n distinct values, and those values are the nth roots of z. To prove this, we write $z = r\exp(i\theta)$, where θ is the principal value of $\mathrm{Arg} z$. Then, in view of definition (3.2.2), Sec. 3.2, of $\mathrm{Log} z$,

$$\exp(\frac{1}{n}\mathrm{Log} z) = \left\{ \exp\left(\frac{1}{n}\ln r + \frac{i(\theta + 2k\pi)}{n} \right) : k \in \mathbf{Z} \right\}.$$

Thus, from the definition of the exponential function, we obtain that

$$\exp(\frac{1}{n}\mathrm{Log} z) = \left\{ \sqrt[n]{r}\exp\left(i\frac{\theta + 2k\pi}{n} \right) : k \in \mathbf{Z} \right\} = z^{1/n}. \tag{3.4.6}$$

This establishes property (3.4.5), which is also valid for every negative integer n too (see Exercise 5).