§2.14. Examples

As pointed out in Sec. 2.13, it is often possible to determine where a given function is analytic by simply recalling various differentiation formulas in Sec. 2.9.

Example 1. The quotient

$$w = f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is evidently analytic throughout the z plane except for the singular points $z = \pm\sqrt{3}$ and $z = \pm i$. The analyticity is due to the existence of familiar differentiation formulas, which need be applied only if the expression for $f'(z)$ is wanted.

When a function is given in terms of its component functions $u(x, y)$ and $v(x, y)$, its analytic can be demonstrated by direct application of the Cauchy-Riemann equations.

Example 2. When

$$w = f(z) = \cosh x \cos y + i \sinh x \sin y,$$

the component functions are

$$u(x, y) = \cosh x \cos y \quad \text{and} \quad v(x, y) = \sinh x \sin y.$$

Because

$$u_x(x, y) = \sinh x \cos y = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -\cosh x \sin y = -v_x(x, y)$$

exist and continuous everywhere, it is clear from Corollary 2.11.2 that f is entire.

Finally, we illustrate how the theorems in the last four sections, in particular the one in Sec. 2.13, can be used to obtain some important properties of analytic functions.

Example 3. Suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

and its conjugate

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

are both analytic in a given domain D. Then $f(z)$ must be constant throughout D.