§6.7. Zeros of Analytic Functions

Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions.

Definition 6.7.1. Suppose that a function f is analytic at a point z_0. If $f(z_0)=0$ and if there is a positive integer m such that

$$f^{(k)}(z_0)=0(k=1,2,\ldots,m-1) \text{ and } f^{(m)}(z_0)\neq 0,$$

then we call the point z_0 a *zero of order* m *of* f. Also, we say that the function has a zero of order m at z_0.

Our first theorem here provides a useful alternative characterization of zeros of order m.

Theorem 6.7.1. *A function* f *has a zero of order* m *at* z_0 *if and only if there is a function* g*, which is analytic and nonzero at* z_0*, such that*

$$f(z)=(z-z_0)^m g(z)\ (|z-z_0|<\varepsilon). \tag{6.7.1}$$

Proof. We assume that expression (6.7.1) holds. Since $g(z)$ is analytic at z_0, it has a Taylor series representation

$$g(z)=g(z_0)+\frac{g'(z_0)}{1!}(z-z_0)+\frac{g''(z_0)}{2!}(z-z_0)^2+\cdots$$

in some neighborhood $|z-z_0|<\varepsilon$ of z_0. Expression (6.7.1) thus takes the form

$$f(z)=g(z_0)(z-z_0)^m+\frac{g'(z_0)}{1!}(z-z_0)^{m+1}+\frac{g''(z_0)}{2!}(z-z_0)^{m+2}+\cdots$$

when $|z-z_0|<\varepsilon$. Since this is actually a Taylor series expansion for $f(z)$, according to Theorem 5.9.1 in Sec. 5.9, it follows that

$$f(z_0)=f'(z_0)=f''(z_0)=\cdots=f^{(m-1)}(z_0)=0 \tag{6.7.2}$$

and that

$$f^{(m)}(z_0)=m!g(z_0)\neq 0. \tag{6.7.3}$$

Hence z_0 is a zero of order m of f.

Conversely, if we assume that f has a zero of order m at z_0, its analyticity at z_0 and the fact that conditions (6.7.2) hold tell us that, in some neighborhood $|z-z_0|<\varepsilon$, there is a Taylor series

$$\begin{aligned} f(z)&=\sum_{n=m}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \\ &=(z-z_0)^m\left[\frac{f^{m}(z_0)}{m!}+\frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)+\frac{f^{(m+2)}(z_0)}{(m+2)!}(z-z_0)^2+\cdots\right] \\ &=(z\text{-}z_0)^m g(z), \end{aligned}$$

where

$$\begin{aligned} g(z)=&\frac{f^{(m)}(z_0)}{m!}+\frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) \\ &+\frac{f^{(m+2)}(z_0)}{(m+2)!}(z-z_0)^2+\cdots\quad(|z-z_0|<\varepsilon). \end{aligned}$$

The convergence of this last series when $|z-z_0|<\varepsilon$ ensures that g is analytic in that neighborhood and, in particular, at z_0 (Sec. 5.8). Moreover,

$$g(z_0) = \frac{f^{(m)}(z_0)}{m!} \neq 0 .$$

This completes the proof of the theorem.

Example. The entire function $f(z) = z(e^z - 1)$ has a zero of order $m = 2$ at the point $z_0 = 0$ since

$$f(0) = f'(0) = 0 \text{ and } f''(0) = 2 \neq 0 .$$

The function g in expression (6.7.1) is defined by

$$g(z) = \begin{cases} (e^z - 1)/z, & \text{when } z \neq 0, \\ 1, & \text{when } z = 0. \end{cases}$$

It is analytic at $z = 0$ and, in fact, entire (see Exercise 4, Sec. 5.9).

Our next theorem tells us that the zeros of a nonzero analytic function are *isolated.*

Theorem 6.7.2. *Suppose that*

(i) f *is analytic at* z_0*;*

(ii) $f(z_0) = 0$ *but* $f(z)$ *is not identically equal to zero in any neighborhood of* z_0.

Then $f(z) \neq 0$ *throughout some* $N^{\circ}(z_0, \varepsilon)$.

Proof. To prove this, let f be as stated and observe that not all of the derivatives of f at z_0 are zero. For, if they were, all of the coefficients in the Taylor series for f about z_0 would be zero; and that would mean that $f(z)$ is identically equal to zero in some neighborhood of z_0. So it is clear from the definition of zeros of order m at the beginning of this section that f must have a zero of some order m at z_0. According to Theorem 6.7.1, then,

$$f(z) = (z - z_0)^m g(z) \tag{6.7.4}$$

where $g(z)$ is analytic and nonzero at z_0.

Now g is continuous, in addition to being nonzero, at z_0 because it is analytic there. Hence, there is some neighborhood $N(z_0, \varepsilon)$ in which equation (6.7.4) holds and in which $g(z) \neq 0$ (See Sec. 2.7). So, $f(z) \neq 0$ in the *throughout some* $N^{\circ}(z_0, \varepsilon)$; and the proof is complete.

Our final theorem here concerns functions with zeros that are not all isolated. It was referred to earlier in Sec. 2.16 and makes an interesting contrast to Theorem 6.7.2 just above.

Theorem 6.7.3. *Suppose that* f *is analytic throughout a neighborhood* N_0 *of* z_0 *and* $f(z_0) = 0$. *If* $f(z) = 0$ *at each point* z *of a domain or line segment containing* z_0 (Fig. 6-8), *then* $f(z) \equiv 0$ *in* N_0.

Proof. We begin the proof with the observation that, under the stated conditions, $f(z) \equiv 0$ in some neighborhood N of z_0. For, otherwise, there would be a deleted neighborhood of z_0 throughout which $f(z) \neq 0$, according to Theorem 6.7.2 above; and that would be inconsistent with the condition that $f(z) = 0$ everywhere in a domain or on a line segment containing z_0.

Since $f(z) \equiv 0$ in the neighborhood N, then, it follows that all of the coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, \ldots)$$

in the Taylor series for $f(z)$ about z_0 must be zero. Thus $f(z) \equiv 0$ in the neighborhood N_0, since Taylor series also represents $f(z)$ in N_0. This completes the proof.

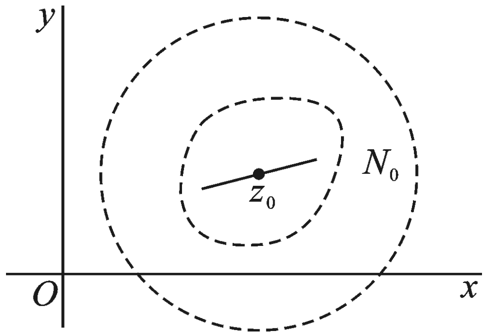

Fig. 6-8