§2.5. Theorems on Limits

We can expedite our treatment of limits by establishing a connection between limits of functions of a complex variable and limits of real-valued functions of two real variables. Since limits of the latter type are studied in calculus, we use their definition and properties freely.

Theorem 2.5.1. *Suppose that* f *is a complex-valued function defined on* $D \subset \mathbf{C}$ *and* $u(x,y) = \operatorname{Re} f(z)$, $v(x,y) = \operatorname{Im} f(z)$, $x_0 = \operatorname{Re} z_0, y_0 = \operatorname{Im} z_0$, $u_0 = \operatorname{Re} w_0, v_0 = \operatorname{Im} w_0$, *then*

$$\lim_{z \to z_0} f(z) = w_0 \tag{2.5.1}$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 . \tag{2.5.2}$$

Theorem 2.5.2. *Suppose that*

$$\lim_{z \to z_0} f(z) = w_0 \quad \textit{and} \quad \lim_{z \to z_0} F(z) = W_0 , \tag{2.53}$$

then

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0 , \tag{2.5.4}$$

$$\lim_{z \to z_0} [f(z)F(z)] = w_0 W_0 ; \tag{2.5.5}$$

and, if in addition, $W_0 \neq 0$, *then*

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} . \tag{2.5.6}$$