## §2.8. Derivatives

**1. Definition of a derivable function**

Let $f$ be a function whose domain of definition contains a neighborhood of a point $z_0$. The *derivative* of $f$ at $z_0$, written $f'(z_0)$, is defined by the equation

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2.8.1}$$

provided this limit exists. The function $f$ is said to be *differentiable* (or *derivable*) at $z_0$ when its derivative at $z_0$ exists.

By expressing the variable $z$ in definition (2.8.1) in terms of the complex variable $\Delta z = z - z_0$, we can write that definition as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{2.8.2}$$

Note that, because $f$ is defined throughout a neighborhood of $z_0$, the number $f(z_0 + \Delta z)$ is always defined for $|\Delta z|$ sufficiently small (Fig. 2-13).

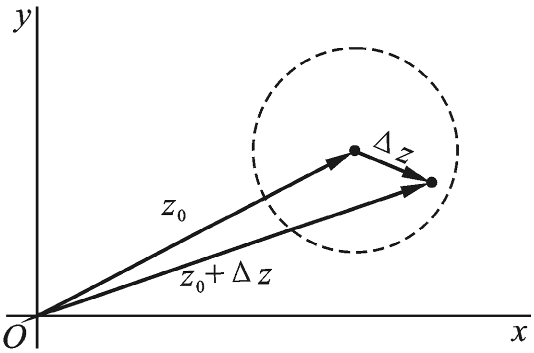


**Fig. 2-13**

When taking form (2.8.2) of the definition of derivative, we often drop the subscript on $z_0$ and introduce the number $\Delta w = f(z + \Delta z) - f(z)$ which denotes the change in the value of $f$ corresponding to a change $\Delta z$ in the point at which $f$ is evaluated. Then, if we write $\frac{dw}{dz}$ for $f'(z)$, equation (2.8.2) becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{2.8.3}$$

**2. Examples**

**Example 1.** Suppose that $f(z) = z^2$. At any point $z$,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z,$$

since $2z + \Delta z$ is a polynomial in $\Delta z$. Hence, $\frac{dw}{dz} = 2z$, *i.e.* $f'(z) = 2z$.

Some times, we write $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$.

**Example 2.** Consider now the function $f(z) = |z|^2$. Here

$$\frac{f(z+h)-f(z)}{h} = \frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)(\overline{z}+\overline{h}) - z\overline{z}}{h} = \overline{z} + \overline{h} + z\frac{\overline{h}}{h}.$$

If the function $f$ is differentiable at a point $z$, then limit

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$$

exists. Take

$$h_n' = \frac{1}{n}, h_n'' = \frac{i}{n},$$

then

$$h_n', h_n'' \to 0 (n \to \infty).$$

See Fig. 2-14.

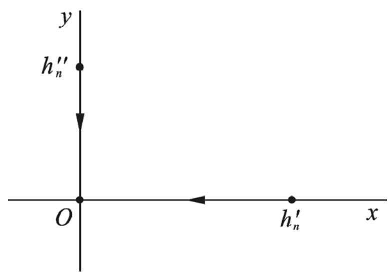


**Fig. 2-14**

Thus, from Theorem 2.4.2 we know that

$$\lim_{n\to\infty}\frac{f(z+h_n')-f(z)}{h_n'} = \lim_{n\to\infty}\frac{f(z+h_n'')-f(z)}{h_n''} = f'(z).$$

Thus, $\overline{z} + z = \overline{z} - z$ and so $z = 0$. This shows that the function $f(z) = |z|^2$ is not differentiable at any nonzero point. To show that $f'(0)$ does, in fact, exist, we need only observe that

$$\frac{f(0+h)-f(0)}{h} = \overline{h} \to 0 (h \to 0).$$

So $f'(0) = 0$.

We conclude, therefore, that $f'(z)$ exists *only* at $z = 0$, its value there being 0.

**Proposition 2.8.1.** *If a function $f$ is differentiable at a point $z_0$, then it is continuous at that point.*