Chapter Ⅷ
Applications of Residues

We turn now to some important applications of the theory of residues, which was developed in the preceding chapter. The applications include evaluation of certain types of definite and improper integrals occurring in *real* analysis and applied mathematics. Considerable attention is also given to a method, based on residues, for locating zeros of functions.

§7.1. Evaluation of Improper Integrals

In calculus, the *improper integral* of a function $f(x)$ over the semi-infinite interval $[0,\infty)$ is defined by means of the equation

$$\int_0^\infty f(x)dx = \lim_{R\to\infty}\int_0^R f(x)dx\,. \tag{7.1.1}$$

When the limit on the right exists, the improper integral is said to *converge* to that limit. If $f(x)$ is continuous for all x, its improper integral over the infinite interval $(-\infty,\infty)$ is defined by writing

$$\int_{-\infty}^\infty f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^0 f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx\,; \tag{7.1.2}$$

and when both of the limits here exist, integral (7.1.2) converges to their sum. Another value that is assigned to integral (7.1.2) is often useful. Namely, the *Cauchy principal value* (P.V.) of integral (7.1.2) is the number

$$P.V.\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_{-R}^R f(x)dx\,, \tag{7.1.3}$$

provided this single limit exists.

If integral (7.1.2) converges, its Cauchy principal value (7.1.3) exists; and that value is the number to which integral (7.1.2) converges. This is because

$$\int_{-R}^R f(x)dx = \int_{-R}^0 f(x)dx + \int_0^R f(x)dx$$

and the limit as $R\to\infty$ of each of the integrals on the right exists when integral (7.1.2) converges. It is *not*, however, always true that integral (7.1.2) converges when its Cauchy principal value exists, as the following example shows.

Example. Observe that

$$P.V.\int_{-\infty}^{\infty} xdx = \lim_{R\to\infty}\int_{-R}^{R} xdx = \lim_{R\to\infty}\frac{x^2}{2}\bigg|_{-R}^{R} = \lim_{R\to\infty} 0 = 0. \tag{7.1.4}$$

On the other hand,

$$\begin{aligned}\int_{-\infty}^{\infty} xdx &= \lim_{R_1\to\infty}\int_{-R_1}^{0} xdx + \lim_{R_2\to\infty}\int_{0}^{R_2} xdx \\ &= \lim_{R_1\to\infty}\frac{x^2}{2}\bigg|_{-R_1}^{0} + \lim_{R_2\to\infty}\frac{x^2}{2}\bigg|_{0}^{R_2} \\ &= -\lim_{R_1\to\infty}\frac{R_1^2}{2} + \lim_{R_2\to\infty}\frac{R_2^2}{2}; \end{aligned} \tag{7.1.5}$$

and since these last two limits do not exist, we find that the improper integral (7.1.5) fails to exist.

But suppose that $y = f(x)(-\infty < x < \infty)$ is an even function, that is, $f(-x) = f(x)$ for all x. The symmetry of the graph of $y = f(x)$ with respect to the y axis enables us to write

$$\int_0^R f(x)dx = \frac{1}{2}\int_{-R}^{R} f(x)dx,$$

and we see that integral (7.1.1) converges to one half the Cauchy principal value (7.1.3) when that value exists. Moreover, since integral (7.1.1) converges and since

$$\int_{-R_1}^{0} f(x)dx = \int_0^{R_1} f(x)dx,$$

integral (7.1.2) converges to twice the value of integral (7.1.1). We have thus shown The following result.

Theorem 7.1.1 *when* $f(x)(-\infty < x < \infty)$ *is even and the Cauchy principal value* (7.1.3) *exists, both of the integrals* (7.1.1) *and* (7.1.2) *converge and*

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x)dx. \tag{7.1.6}$$

We now describe a method involving residues, to be illustrated in the next section, that is often used to evaluate improper integrals of even rational functions $f(x) = p(x)/q(x)$, where $f(-x)$ is equal to $f(x)$ and where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. We agree that $q(z)$ has no real zeros but has at least one zero above the real axis.

The method begins with the identification of all of the distinct zeros of the polynomial $q(z)$ that lie above the real axis. They are, of course, finite in number (see Sec. 4.14) and may be labeled $z_1, z_2, \ldots, z_n$, where n is less than or equal to the degree of $q(z)$. We then integrate the quotient

$$f(z)=\frac{p(z)}{q(z)} \tag{7.1.7}$$

around the positively oriented boundary of the semicircular region shown in Fig. 7-1. That simple closed contour consists of the segment of the real axis from $z=-R$ to $z=R$ and the top half of the circle $|z|=R$, described counterclockwise and denoted by C_R. It is understood that the positive number R is large enough that the points $z_1, z_2, \ldots, z_n$ all lie inside the closed path.

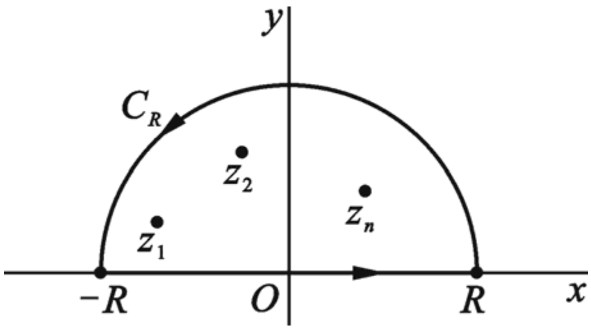

Fig. 7-1

The Cauchy residue theorem tells us that

$$\int_{-R}^{R} f(x)dx+\int_{C_R} f(z)dz=2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z).$$

That is,

$$\int_{-R}^{R} f(x)dx=2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z)-\int_{C_R} f(z)dz. \tag{7.1.8}$$

Now we arrive at the following conclusion.

Theorem 7.1.2. *Let a function* f *be given by* (7.1.7) *and the points* $z_1, z_2, \ldots, z_n$ *have the described property above. If the limit*

$$\lim_{R\to\infty}\int_{C_R} f(z)dz=0,$$

then

$$P.V.\int_{-\infty}^{\infty} f(x)dx=2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z). \tag{7.1.9}$$

If $f(x)$*, in addition, is even, then*

$$\int_{-\infty}^{\infty} f(x)dx=2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z) \tag{7.1.10}$$

and

$$\int_{0}^{\infty} f(x)dx=\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z). \tag{7.1.11}$$