§8.2. Unilateral Functions

In this section, we will discuss a very important class of conformal mappings, called unilateral functions.

Definition 8.2.1. Let f be an analytic functiuon on a domain D. If it is an injection on D, then we call f a *unilateral function* on D.

To discuss the properties of unilateral functions, we need the following lemma.

Lemma 8.2.1. *Suppose that a function f is analytic on a neighbordood $N(z_0,R)$ of a point z_0 such that*

$$w_0 = f(z_0), f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0. \quad (8.1.6.)$$

Then there is a positive number $r < R$ such that $\forall 0 < \varepsilon < r, \exists \delta > 0$ satisfing

(1) *The point z_0 is the unique zero of $f(z) - w_0$ in $N(z_0,\varepsilon)$ and it is of order m*

(2) *For each fixed point $w \in N^\circ(w_0,\delta)$, the function $f(z) - w$ has exactly m zeros counting multiplicity, in the deleted neighborhood $N^\circ(z_0,\varepsilon)$ and these zeros are all of order* 1.

Proof. Since f is not a constant in $N(z_0,r)$, $f'(z)$ is not identically zeron in $N(z_0,r)$. Thus, there is a positive number $r < R$ such that functions $f(z) - w_0$ and $f'(z)$ have no zeros in $\overline{D} = \overline{N(z_0,r)}$ except $z = z_0$. Thus, for each $0 < \varepsilon < r$, it is clear that z_0 is the unique zero of $f(z) - w_0$ in $N(z_0,\varepsilon)$ and it is of order m. Clearly,

$$\delta := \min_{|z-z_0|=\varepsilon} | f(z) - w_0 | > 0.$$

Let $w \in N^\circ(w_0,\delta)$, we write

$$f(z) - w = (f(z) - w_0) + (w_0 - w).$$

Since $| f(z) - w_0 | \geq \delta > | w_0 - w |$ when $| z - z_0 | = \varepsilon$, it follows from the Rouche's theorem that the function $f(z) - w$ has just m zeros in $N^\circ(z_0,\varepsilon)$. Let z_1 be any zero of $f(z) - w$ in $N^\circ(z_0,\varepsilon)$, then

$$\left(f(z) - w\right)'\Big|_{z=z_1} = f'(z_1) \neq 0.$$

Thus, z_1 must be a zero of order 1 of $f(z) - w$. The proof is completed.

Theorem 8.2.1. *Every unilateral function f on a domain D is a conformal mapping on that domain.*

Proof. Suppose that $f'(z_0) = 0$ for some $z_0 \in D$, then there exists a positive integer $m > 1$ such that

$$f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0.$$

Take an $R > 0$ such that $N(z_0,R) \subset D$. From Lemma 8.2.1, there is a positive number $r < R$ such that $\forall 0 < \varepsilon < r, \exists \delta > 0$ so that for each fixed point $w \in N^\circ(w_0,\delta)$, the function $f(z) - w$ has exactly m zeros counting multiplicity in the deleted neighborhood $N^\circ(z_0,\varepsilon)$ and they are all of order 1. Since $m > 1$, for each fixed point $w \in N^\circ(w_0,\delta)$, there are two distinct points z_1, z_2 in $N^\circ(z_0,\varepsilon)$ such that $f(z_1) = f(z_2) = w$. This contradicts the assumption that f is unilateral in D. This proves that f is conformal in D and completes the proof.

Note that a conformal mapping is not necessarily unilateral. For example, the function $f(z) = e^z$ is conformal throughout the plane, but it is not unilateral since it is $2\pi i$-periodic. However, we have the following result.

Theorem 8.2.2. *If a function* f *is conformal at a point* z_0*, then it is unilateral on some neighborhood of* z_0.

Proof. Since $f'(z_0) \neq 0$, using Lemma 8.2.1 for $m = 1$, we know that there exists a neighborhood $N(z_0, \varepsilon)$ of z_0 and a neighborhood $N(w_0, \delta)$ of w_0 such that z_0 is the unique zero of $f(z) - w_0$ in $N(z_0, \varepsilon)$ and it is of order 1, and for each fixed point $w \in N^{\circ}(w_0, \delta)$, the function $f(z) - w$ has exactly 1 zero in the deleted neighborhood $N^{\circ}(z_0, \varepsilon)$. From the continuity of $f(z) - w$, we can choose a positive number $\mu < \varepsilon$ such that

$$f(N(z_0, \mu)) \subset N(w_0, \delta).$$

If f is not unilateral on $N(z_0, \mu)$, then there are two distinct points $z_1, z_2 \in N(z_0, \mu)$ such that $f(z_1) = f(z_2) := w$. Thus, $z_1, z_2 \neq z_0, w \neq w_0$ since z_0 is the unique zero of $f(z) - w_0$ in $N(z_0, \varepsilon)$. We see that the function $f(z) - w$ has two distinct zeros

$$z_1, z_2 \in N^{\circ}(z_0, \mu) \subset N^{\circ}(z_0, \varepsilon).$$

Since $w \in N^{\circ}(w_0, \delta)$, this is impossible. Thus, f is unilateral on $N(z_0, \mu)$. The proof is completed.

Theorem 8.2.3. *Let* f *be a nonconstant analytic function on a domain* D*, then* $f(D)$ *is a domain.*

Proof. Let $w_0 \in f(D)$, then $\exists z_0 \in D$ such that $w_0 = f(z_0)$. Since D is open, there a neighborhood $N(z_0, R) \subset D$. Since f is analytic and nonconstant in D, there is a positive integer m such that

$$f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0.$$

From Lemma 8.2.1, there is a positive number $r < R$ such that $\forall 0 < \varepsilon < r, \exists \delta > 0$ so that for each fixed point $w \in N^{\circ}(w_0, \delta)$, the function $f(z) - w$ has exactly m zeros counting multiplicity in the deleted neighborhood $N^{\circ}(z_0, \varepsilon)$. Thus,

$$N(w_0, \delta) \subset f(N(z_0, \varepsilon)) \subset D.$$

Hence, w_0 is an interior point of $f(D)$ and so $f(D)$ is open.

For points $w_1, w_2 \in f(D)$, take $z_1, z_2 \in D$ with $f(z_k) = w_k (k = 1,2)$. Since D is connected, there is a polygonal line

$$L : z = z(t)(a \leq t \leq b)$$

contained in D such that $z(a) = z_1, z(b) = z_2$. Since f has conitinuous derivative in D, we get a piecewise smooth curve

$$\Gamma : w = w(t) = f(z(t))(a \leq t \leq b)$$

contained in $f(D)$ such that

$$w(a) = w_1, w(b) = w_2.$$

The compactness of the Γ enable us to find a polygonal line Γ' contained in $f(D)$ and jioning w_1 and w_2. This proves that $f(D)$ is a domain in the w-plane. The proof is completed.