## §6.3. Using a Single Residue

If the function $f$ in Cauchy's residue theorem (Sec. 6.2) is, in addition, analytic at each point of the finite plane exterior to $C$, it is sometimes more efficient to evaluate the integral of $f$ around $C$ by finding a *single* residue of a certain related function. We present the method as a theorem.

**Theorem 6.3.1.** *If a function $f$ is analytic everywhere in the complex plane except for a finite number of singular points interior to a positively oriented simple closed path $C$, then*

$$\int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{6.3.1}$$

**Proof.** We begin the derivation of expression (6.3.1) by constructing a circle $|z| = R_1$ as in Fig. 6-7. Then if $C_0$ denotes a positively oriented circle $|z| = R_0$, where $R_0 > R_1$, we know from Laurent's theorem (Sec. 5.4) that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad (R_1 < |z| < \infty), \tag{6.3.2}$$

where

$$c_n = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)dz}{z^{n+1}} \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{6.3.3}$$

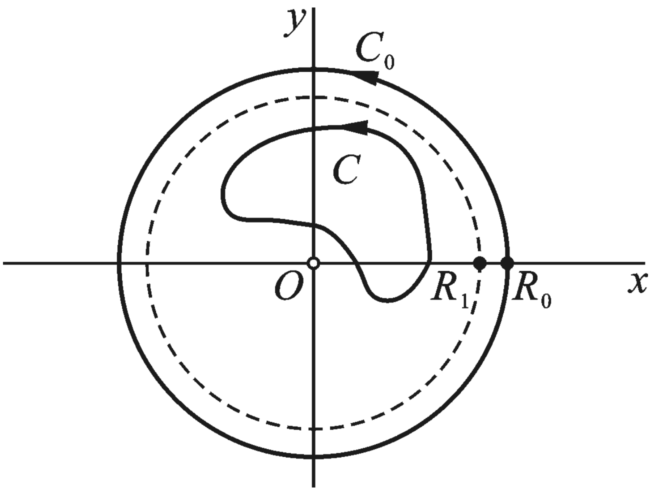


**Fig. 6-7**

By writing $n = -1$ in expression (6.3.3), we find that

$$\int_{C_0} f(z)dz = 2\pi i c_{-1}. \tag{6.3.4}$$

Observe that, since the condition of validity with representation (6.3.2) is not of the type $0 < |z| < R_2$, the coefficient $c_{-1}$ is *not* necessarily the residue of $f$ at the point $z = 0$, which may not even be a singular point of $f$. But, if we replace $z$ by $1/z$ in representation (6.3.2) and its condition of validity, we see that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} \quad \left(0 < |z| < \frac{1}{R_1}\right)$$

and hence

$$c_{-1} = \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]. \tag{6.3.5}$$

Then, in view of equations (6.3.4) and (6.3.5),

$$\int_{C_0} f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

Finally, since $f$ is analytic throughout the closed region bounded by $C$ and $C_0$, the principle of deformation of paths (Corollary 4.11.2, Sec. 4.11) yields the desired result (6.3.1).

**Example 1.** In the example in Sec. 6.2, we evaluated the integral of

$$f(z) = \frac{5z-2}{z(z-1)}$$

around the circle $|z| = 2$, described counterclockwise, by finding the residues of $f(z)$ at $z = 0$ and $z = 1$. Since when $0 < |z| < 1$,

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{5-2z}{z(1-z)} = \frac{5-2z}{z} \cdot \frac{1}{1-z}$$

$$= \left(\frac{5}{z} - 2\right)(1 + z + z^2 + \cdots)$$

$$= \frac{5}{z} + 3 + 3z + \cdots,$$

we see from Theorem 6.3.1 that, where the desired residue is 5,

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i \times 5 = 10\pi i,$$

where $C$ is the circle in question. This is just the result obtained in Example 1 in Sec. 6.2.