§2.12. Polar Coordinates

Assuming that $z_0 \neq 0$, we shall in this section use the coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta \tag{2.12.1}$$

to restate Theorem 2.11.1 in Sec. 2.11 in polar coordinates.

1. Theorem

Theorem 2.12.1. *Let the function*

$$f(z) = u(r,\theta) + iv(r,\theta)(z = re^{i\theta})$$

be defined on some ε *neighborhood of a nonzero point* $z_0 = r_0 e^{i\theta_0}$. *Then* f *is differentiable at* (r_0, θ_0) *if and only if* u *and* v *are differentiable at* (r_0, θ_0) *and satisfy the polar form*

$$ru_r(r_0, \theta_0) = v_\theta(r_0, \theta_0), \quad u_\theta(r_0, \theta_0) = -rv_r(r_0, \theta_0)$$

of the Cauchy-Riemann equations. In this case, the derivative $f'(z_0)$ *can be written*

$$f'(z_0) = e^{-i\theta_0}[u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)], \tag{2.12.7}$$

(see Exercise 8).

2. Examples

Example 1. Consider the function

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) \quad (z \neq 0). \tag{2.12.8}$$

Since

$$u(r,\theta) = \frac{\cos\theta}{r} \text{ and } v(r,\theta) = \frac{\sin\theta}{r},$$

the conditions in Theorem 2.12.1 are satisfied at every nonzero point $z = re^{i\theta}$ in the plane. In particular, the Cauchy-Riemann equations

$$ru_r(r,\theta) = -\frac{\cos\theta}{r} = v_\theta(r,\theta) \text{ and } u_\theta(r,\theta) = -\frac{\sin\theta}{r} = -rv_r(r,\theta)$$

are satisfied. Hence, the derivative of f exists when $z \neq 0$; and, according to expression (2.12.7),

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -e^{i\theta}\frac{e^{-i\theta}}{r^2} = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}.$$

Example 2. Theorem 2.12.1 can be used to show that, when α is a fixed real number, the function

$$f(z) = \sqrt[3]{r}e^{i\theta/3} \quad (r > 0, \alpha < \theta < \alpha + 2\pi) \tag{2.12.9}$$

has a derivative everywhere in its domain of definition. Here

$$u(r,\theta) = \sqrt[3]{r}\cos\frac{\theta}{3} \text{ and } v(r,\theta) = \sqrt[3]{r}\sin\frac{\theta}{3}.$$

Since

$$ru_r(r,\theta) = \frac{\sqrt[3]{r}}{3}\cos\frac{\theta}{3} = v_\theta(r,\theta) \text{ and } u_\theta(r,\theta) = -\frac{\sqrt[3]{r}}{3}\sin\frac{\theta}{3} = -rv_r(r,\theta)$$

and the other conditions in Theorem 2.12.1 are satisfied, the derivative $f'(z)$ exists at each point where $f(z)$ is defined. Furthermore, expression (2.12.7) tells us that

$$f'(z) = e^{-i\theta}\left[\frac{1}{3(\sqrt[3]{r})^2}\cos\frac{\theta}{3} + i\frac{1}{3(\sqrt[3]{r})^2}\sin\frac{\theta}{3}\right],$$

hence

$$f'(z) = \frac{e^{-i\theta}}{3(\sqrt[3]{r})^2} e^{i\theta/3} = \frac{1}{3(\sqrt[3]{r}e^{i\theta/3})^2} = \frac{1}{3[f(z)]^2}.$$

Note that when a specific point z is taken in the domain of definition of f, the value $f(z)$ is one value of $z^{1/3}$ (see Sec. 2.1). Hence this last expression for $f'(z)$ can be put in the form

$$\frac{d}{dz} z^{1/3} = \frac{1}{3(z^{1/3})^2}$$

when that value is taken. Derivatives of such power functions will be elaborated on in Chap. 3 (Sec. 2.22).