§2.10. Cauchy-Riemann Equations

In this section, we obtain a pair of equations that the first-order partial derivatives of the component functions u and v of a function

$$f(z) = u(x, y) + iv(x, y) \tag{2.10.1}$$

must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists there. We also show how to express $f'(z_0)$ in terms of those partial derivatives.

1. Cauchy-Riemann equations

Suppose that $f(z) = u(x, y) + iv(x, y)$ *and that* $f'(z_0)$ *exists where* $z_0 = x_0 + iy_0$. *Then*

(1) *The functions* u *and* v *must be differentiable at* (x_0, y_0);

(2) *The Cauchy-Riemann equations*

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \tag{2.10.2}$$

are satisfied;

(3) *The derivative* $f'(z_0)$ *can be written as*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -i[u_y(x_0, y_0) + iv_y(x_0, y_0)]. \tag{2.10.3}$$

2. Examples

Example 1. In Example 1, Sec. 2.8, we showed that the function

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

is differentiable everywhere and that $f'(z) = 2z$. To verify that the Cauchy-Riemann equations are satisfied everywhere, we note that

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Thus

$$u_x = 2x = v_y, \quad u_y = -2x = -v_x.$$

Moreover, according to equation (2.10.8),

$$f'(z) = 2x + i2y = 2(x + iy) = 2z.$$

Since the Cauchy-Riemann equations are *necessary* conditions for the existence of the derivative of a function f at a point z_0, they can often be used to locate points at which f does *not* have a derivative.

Example 2. When $f(z) = |z|^2$, we have

$$u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0.$$

If the Cauchy-Riemann equations are satisfied at a point (x, y), it follows that $2x = 0$ and $2y = 0$, that is $x = y = 0$. Consequently, $f'(z)$ does not exist at any nonzero point, as we already know from Example 2 in Sec. 2.8. Note that the above theorem does not ensure the existence of $f'(0)$. The theorem in the next section will, however, do this.