§6.11. Reflection Principle

The theorem in this section concerns the fact that some analytic functions possess the property that $\overline{f(z)} = f(\bar{z})$ for all points z in a domain D (in this case, we say that f is *conjugate preserving on* D), while others do not. We note, for example, that $z+1$ and z^2 have that property when D is the entire finite plane; but the same is not true of $z+i$ and iz^2. Generally, every polynomial with real coefficients is conjugate-preserving. The following theorem, which is known as the reflection principle, provides a way of predicting when $\overline{f(z)} = f(\bar{z})$.

Theorem 6.11.1(Reflection Principle). *If* f *is analytic on a domain* D *which contains a segment* L *of the* x*-axis and satisfies* $z \in D \Leftrightarrow \bar{z} \in D$, *then* f *is conjugate preserving on* D, i.e.

$$\overline{f(z)} = f(\bar{z}), \forall z \in D \tag{6.11.1}$$

if and only if $f(x)$ *is real for each point* x *in* L.

Proof. *Sufficiency.* We assume that $f(x)$ is real at point x on the segment. Once we show that the function

$$F(z) = \overline{f(\bar{z})} \tag{6.11.2}$$

is analytic in D, we shall use it to obtain equation (6.11.1). To establish the analyticity of $F(z)$, we write

$$f(z) = u(x,y) + iv(x,y), \qquad F(z) = U(x,y) + iV(x,y).$$

Since

$$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y), \quad \forall (x,y) \in D, \tag{6.11.3}$$

the components of $f(z)$ and $F(z)$ are related by the equations

$$U(x,y) = u(x,-y) \text{ and } V(x,y) = -v(x,-y), \ \forall (x,y) \in D. \tag{6.11.4}$$

Now, because f is analytic in D, u and v satisfy the Cauchy-Riemann equations

$$u_x(x,-y) = v_y(x,-y), \quad u_y(x,-y) = -v_x(x,-y), \ \forall (x,y) \in D. \tag{6.11.5}$$

Furthermore, in view of equations (6.11.4),

$$U_x(x,y) = u_x(x,-y), \qquad V_y(x,y) == v_y(x,-y), \ \forall (x,y) \in D;$$

and it follows from these and the first of equations (6.11.5) that

$$U_x(x,y) = V_y(x,y), \ \forall (x,y) \in D.$$

Similarly,

$$U_y(x,y) = -u_y(x,-y) = v_x(x,-y) = -V_x(x,y), \ \forall (x,y) \in D.$$

Thus, $U(x,y)$ and $V(x,y)$ are now shown to satisfy the Cauchy-Riemann equations. Since u and v are differentiable in D, so are U and V. We find that the function $F(z)$ is analytic in D. Moreover, since $f(x)$ is real on the segment of the real axis lying in D, $v(x,0) = 0$ on that segment; and, in view of equations (6.11.4), this means that

$$F(x) = U(x,0) + iV(x,0) = u(x,0) - iv(x,0) = u(x,0) = f(x).$$

That is,

$$F(z) = f(z) \tag{6.11.6}$$

at each point on the segment. It follows from Theorem 6.8.1 that equation (6.11.6) actually holds throughout D. Because of definition (6.11.2) of the function $F(z)$, then,

$$\overline{f(\bar{z})} = f(z) \text{ for all } z \in D, \tag{6.11.7}$$

which is the same as equation (6.11.1).

Necessity. We assume that equation (6.11.1) holds and note that, in view of expression (6.11.3), we have

$$u(x,-y) - iv(x,-y) = u(x,y) + iv(x,y),\ \forall (x,y) \in D.$$

In particular, if $(x,0)$ is a point on the segment L, then

$$u(x,0) - iv(x,0) = u(x,0) + iv(x,0);$$

thus $v(x,0) = 0$. Hence $f(x)$ is real on the segment L.

This completes the proof.

Examples. Just prior to the statement of the theorem, we noted that

$$\overline{z+1} = \overline{z} + 1 \text{ and } \overline{z^2} = \overline{z}^2$$

for all z in the finite plane. Theorem 6.11.1 tells us that this is true, since $x+1$ and x^2 are real when x is real. We also noted that $z+i$ and iz^2 do not the reflection property throughout the plane, and we now know that this is because $x+i$ and ix^2 are not real when x is real.