§4.6. Upper Bounds for Integrals

In this section, we will give an upper bound for the modulus of a path integral, wich is useful in estimation of an integral.

Theroem 4.6.1 *If a function* f *is contionuous on a countour* C *and* M *is a constant such that* $|f(z)| \le M\,(\forall z \in C)$, *then*

$$\left|\int_C f(z)dz\right| \le ML, \tag{4.6.1}$$

where L *denotes the length of* C.

Proof. Let $z = z(t) \quad (a \le t \le b)$ be the parametric representation of C. Fisrt, we know from definition (4.4.2), Sec. 4.4, and inequality (4.2.5) in Sec. 4.2 that

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f[z(t)]z'(t)dt\right| \le \int_a^b |f[z(t)]||z'(t)|\,dt.$$

Since $|f(z)| \le M\,(\forall z \in C)$, we have

$$\left|\int_C f(z)dz\right| \le M\int_a^b |z'(t)|\,dt = ML.$$

This completes the proof.

Note that since all of the paths of integration to be considered here are paths and the integrands are piecewise continuous functions defined on those paths, a number M such as the one appearing in inequality (4.6.1) will always exist.

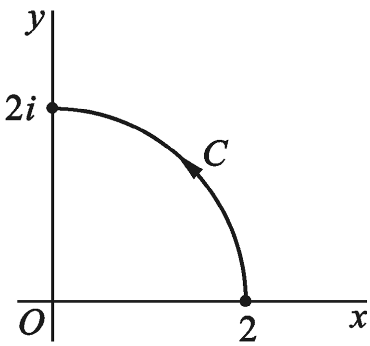

Fig. 4-11

Example 1. Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant (Fig. 4-11). Show that

$$\left|\int_C \frac{z+4}{z^3-1}dz\right| \le \frac{6\pi}{7}. \tag{4.6.2}$$

Proof. Let z be any point on C. Then $|z| = 2$ and so

$$|z+4| \le |z| + 4 = 6 \quad \text{and} \quad |z^3 - 1| \ge ||z|^3 - 1| = 7.$$

Thus, when z lies on C, we have

$$\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \le \frac{6}{7}.$$

Writing $M = 6/7$ and observing that $L = \pi$ is the length of C, we may now use inequality (4.6.1) to obtain inequality (4.6.2). This completes the proof.

Example 2. Suppose that C_R is the semicircular path

$$z = Re^{i\theta} \quad (0 \le \theta \le \pi),$$

and $z^{1/2}$ denotes the branch

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \quad \left(r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2} \right)$$

of the square root function. Show that (Fig. 4-12)

$$\lim_{R\to\infty} \int_{C_R} \frac{z^{1/2}}{z^2+1} dz = 0 . \tag{4.6.3}$$

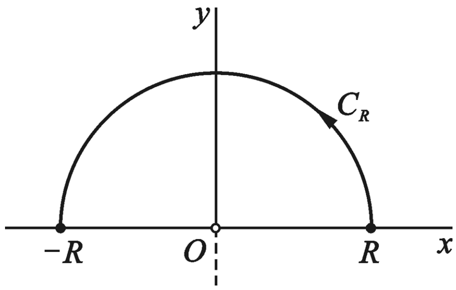

Fig. 4-12

Proof. When $R > 1$, for every $z \in C_R$, we have $|z| = R > 1$, and

$$|z^{1/2}| = |\sqrt{R}e^{i\theta/2}| = \sqrt{R}$$

and then

$$|z^2+1| \geq ||z^2|-1| = R^2 - 1 .$$

Consequently, at points on C_R, we have

$$\left| \frac{z^{1/2}}{z^2+1} \right| \leq M_R \text{, where } M_R = \frac{\sqrt{R}}{R^2-1} .$$

Since the length of C_R is the number $L = \pi R$, it follows from inequality (4.6.1) that

$$\left| \int_{C_R} \frac{z^{1/2}}{z^2+1} dz \right| \leq M_R L .$$

But

$$M_R L = \frac{\pi R\sqrt{R}}{R^2-1} = \frac{\pi/\sqrt{R}}{1-(1/R^2)} \to 0 (R \to \infty) .$$

Limit (4.6.3) is, therefore, established. This completes the proof.